From Doubled Chern-Simons-Maxwell Lattice Gauge Theory to Extensions of the Toric Code

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September 11, 2015

Abstract

We regularize compact and non-compact Abelian Chern-Simons-Maxwell theories on a spatial lattice using the Hamiltonian formulation. We consider a doubled theory with gauge fields living on a lattice and its dual lattice. The Hilbert space of the theory is a product of local Hilbert spaces, each associated with a link and the corresponding dual link. The two electric field operators associated with the link-pair do not commute. In the non-compact case with gauge group $\mathbb{R}$, each local Hilbert space is analogous to the one of a charged "particle" moving in the link-pair group space $\mathbb{R}^2$ in a constant "magnetic" background field. In the compact case, the link-pair group space is a torus $U(1)^2$ threaded by k units of quantized "magnetic" flux, with k being the level of the Chern-Simons theory. The holonomies of the torus $U(1)^2$ give rise to two self-adjoint extension parameters, which form two non-dynamical background lattice gauge fields that explicitly break the manifest gauge symmetry from $U(1)$ to $\mathbb{Z}(k)$. The local Hilbert space of a link-pair then decomposes into representations of a magnetic translation group. In the pure Chern-Simons limit of a large "photon" mass, this results in a $\mathbb{Z}(k)$-symmetric variant of Kitaev's toric code, self-adjointly extended by the two non-dynamical background lattice gauge fields. Electric charges on the original lattice and on the dual lattice obey mutually anyonic statistics with the statistics angle $\frac{2\pi}{k}$. Non-Abelian $U(k)$ Berry gauge fields that arise from the self-adjoint extension parameters may be interesting in the context of quantum information processing.

1 Introduction

Gauge theories in two spatial dimensions may contain a Chern-Simons term [1–5] which explicitly breaks parity and time-reversal symmetry. In these theories the gauge field acquires a mass and the charged particles obey fractional anyonic statistics [6–9]. Non-Abelian Chern-Simons theories have intricate relations to knot theory, the Jones polynomials [10], and to 2-dimensional conformal field theories [11, 12]. Abelian Chern-Simons theories have been used to facilitate bosonization or Fermi-Bose transmutation in $(2+1)$ dimensions [13–16]. Furthermore, Chern-Simons gauge theories are of central importance in the context of the fractional quantum Hall effect [17–19] and other condensed matter systems [20–26]. They also play an important role for topological quantum computation [27–32]. By braiding world-lines of anyonic quasi-particles, one can accumulate appropriate non-Abelian Berry phases [33, 34], which can encode quantum information. When the anyons obey a sufficiently complex version of non-Abelian braid statistics, they can be employed to realize the quantum gates that are sufficient to realize universal quantum computation [35–37]. The idea of topological quantum computation is attractive because the quantum information is then naturally protected from decoherence by the topological nature of the non-Abelian Berry phases. In particular, information is not stored locally but is distributed throughout the entire system. The toric code is a $(2+1)$-d $\mathbb{Z}(2)$ lattice gauge theory which can be used as a topologically protected storage device for quantum information [27]. This theory has charges and dual charges with mutually anyonic statistics.

In this paper, we derive extensions of the toric code from a doubled compact lattice Chern-Simons-Maxwell theory with Abelian gauge group $U(1)$. Similar to fermions, topologically massive gauge fields also suffer from a lattice doubling problem. Here we are not trying to circumvent this problem but work with a lattice gauge field and an independent gauge field associated with the dual lattice. The fundamental gauge degrees of freedom are then associated with a cross formed by a link and its corresponding dual link. The Chern-Simons term couples the two lattices and implies that the canonically conjugate momenta (i.e. the electric field strengths) of the original and the dual gauge field do not commute. In ordinary lattice gauge theories [38–41] the field algebra is link-based. This framework has also been used in studies of Chern-Simons gauge theories on the lattice [14, 42–45]. In our lattice formulation of a doubled Chern-Simons gauge theories, on the other hand, the field algebra is cross-based. Such a system was already investigated in [46, 47]. Here we concentrate on the relation of this theory to the toric code.

In ordinary lattice gauge theory with a link-based field algebra every link has a "mechanical" analog. It behaves like a "particle" moving in the group space. For example, the dynamics of a link variable in a compact Abelian $U(1)$ lattice gauge theory is analogous to the one of a quantum mechanical particle moving on a circle. Similarly, in our unconventional lattice gauge theory with a cross-based field algebra, the "mechanical" analog of each cross is a charged "particle" moving on a 2-dimensional group space torus $U(1)^2$ threaded by an abstract "magnetic" field [48–50]. The Dirac quantization condition for the abstract "magnetic" flux then implies the quantization of the level k — the prefactor of the Chern-Simons term. Interestingly, the corresponding cross-based Hamiltonian has two self-adjoint extension parameters, which naturally enter the quantum theory as external parameters, while the classical theory is insensitive to these parameters. Remarkably, the self-adjoint extension parameters (which are associated with the links and the dual links) themselves form two non-dynamical $U(1)$ lattice gauge fields. This reduces the manifest gauge symmetry of the quantum theory from $U(1)$ to $\mathbb{Z}(k)$. This quantum mechanical breaking of the gauge symmetry could be called an "anomaly". However, it is more like the explicit breaking of CP invariance caused by a non-zero θ-vacuum angle in 4-dimensional non-Abelian gauge theories. It is intriguing that the quantized doubled Chern-Simons-Maxwell lattice gauge theory dynamically reduces its manifest gauge symmetry from $U(1)$ to the $\mathbb{Z}(k)$ gauge group of an extended toric code [51, 52]. The full dynamics reduces to the one of the toric code in the pure Chern-Simons limit of infinite "photon" mass.

The main purpose of this paper is to perform a detailed mathematical derivation of an extended version of the $\mathbb{Z}(k)$ variant of the toric code as a limit of doubled $U(1)$ Chern-Simons-Maxwell lattice gauge theory. We put particular emphasis on the role played by the self-adjoint extension parameters of the cross-based Hamiltonian. This embeds the toric code in a wider theoretical framework and thus provides a broader perspective on quantum information processing. In particular, it would be interesting to investigate whether Berry phases that arise from adiabatic changes of the external self-adjoint extension parameters can be utilized for quantum information processing. In this paper, we do not yet address these questions. We also view the present paper as a first step towards similar studies in the context of non-Abelian Chern-Simons theories on the lattice. Variants of the toric code with a discrete non-Abelian gauge group [53], which are discussed in the context of topological quantum computation, may be related to Chern-Simons gauge theories with continuous gauge groups in a similar manner.

The rest of the paper is organized as follows. In Section 2 we discuss Chern-Simons-Maxwell gauge theory of a single Abelian gauge field in the continuum, while in Section 3 we investigate the doubled theory with two Abelian gauge fields. In Section 4 we regularize this theory on a lattice and its dual lattice, with a cross-based field algebra using non-compact Abelian lattice gauge fields. In particular, we discuss the mutual anyonic statistics of the charges moving on the original and the dual lattice. In Section 5 we turn to compact Abelian gauge fields, which leads to the quantization of the level k as well as to the generation of a non-dynamical background lattice gauge field formed by the self-adjoint extension parameters of the cross-based Hamiltonian. In the limit of a large "photon" mass this theory reduces to the toric code. Finally, Section 6 contains our conclusions.

2 Chern-Simons-Maxwell Theory of a Single Abelian Gauge Field in the Continuum

In this section we investigate Abelian Chern-Simons-Maxwell theory in the continuum. After considering a single Abelian gauge field, in the next section we will study a doubled theory which will turn out to arise in the continuum limit of the lattice theory that we discuss later.

2.1 Lagrangian and Hamiltonian

Let us consider a (2+1)-d Abelian gauge field $A_\mu(x)$ with the Chern-Simons-Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{4\pi} \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho, \tag{2.1}$$

where e is the electric charge and the metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$. The field strength is given by $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. Without loss of generality, in the following we assume that the prefactor of the Chern-Simons density $k > 0$. It is interesting to note that the Chern-Simons density is not gauge invariant. However, its variation under a gauge transformation

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \chi(x), \tag{2.2}$$

is a total divergence

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = -\partial^\mu \left(\frac{k}{4\pi} \epsilon_{\mu\nu\rho} \chi \partial^\nu A^\rho \right). \tag{2.3}$$

It should be noted that the Chern-Simons term explicitly breaks both time-reversal and parity (i.e. the reflection on a spatial axis). In order to derive the corresponding Hamiltonian, we now fix to the temporal gauge $A_0(x) = 0$, which implies

$$\mathcal{L} = \frac{1}{2e^2} \dot{A}_i^2 - \frac{1}{2e^2} (\epsilon_{ij} \partial_i A_j)^2 + \frac{k}{4\pi} \epsilon_{ij} \dot{A}_i A_j. \tag{2.4}$$

The momentum canonically conjugate to A_i is then given by

$$\Pi_i(x) = \frac{\delta\mathcal{L}}{\delta \dot{A}_i(x)} = \frac{1}{e^2} \dot{A}_i(x) + \frac{k}{4\pi} \epsilon_{ij} A_j(x), \tag{2.5}$$

such that the classical Hamilton density takes the form

$$\begin{aligned} \mathcal{H} &= \Pi_i \dot{A}_i - \mathcal{L} = \frac{1}{2e^2} \dot{A}_i^2 + \frac{1}{2e^2} (\epsilon_{ij} \partial_i A_j)^2 \\ &= \frac{e^2}{2} \left(\Pi_i - \frac{k}{4\pi} \epsilon_{ij} A_j \right)^2 + \frac{1}{2e^2} (\epsilon_{ij} \partial_i A_j)^2 = \frac{e^2}{2} E_i^2 + \frac{1}{2e^2} B^2. \end{aligned} \tag{2.6}$$

Here we have identified the electric and magnetic fields as

$$E_i(x) = \Pi_i(x) - \frac{k}{4\pi} \epsilon_{ij} A_j(x), \quad B(x) = \partial_1 A_2(x) - \partial_2 A_1(x). \tag{2.7}$$

2.2 Solutions of the Classical Equations of Motion

Before we quantize the theory, we consider its classical equations of motion

$$\partial^\mu F_{\mu\nu}(x) + \frac{ke^2}{4\pi}\epsilon_{\nu\rho\sigma}F^{\rho\sigma}(x) = 0, \tag{2.8}$$

which in components take the form

$$\dot{E}_i(x) + \frac{1}{e^2}\epsilon_{ij}\partial_j B(x) + \frac{ke^2}{2\pi}\epsilon_{ij}E_j(x) = 0, \quad \partial_i E_i(x) + \frac{k}{2\pi}B(x) = 0. \tag{2.9}$$

The second equation is the Gauss law. In particular, the magnetic field acts like a charge density. In addition, the field strength obeys the Bianchi identity

$$\epsilon_{\mu\nu\rho}\partial^\mu F^{\nu\rho}(x) = 0. \tag{2.10}$$

Let us make the simple plane wave ansatz

$$E_i(x) = C_i \cos(\vec{p}\cdot\vec{x} - \omega t) + D_i \sin(\vec{p}\cdot\vec{x} - \omega t). \tag{2.11}$$

Inserting this in the equations of motion, one obtains

$$\omega = \sqrt{M^2 + p^2}, \quad M = \frac{ke^2}{2\pi}, \quad C_i = c\epsilon_{ij}p_j - d\frac{M}{\omega}p_i, \quad D_i = d\epsilon_{ij}p_j + c\frac{M}{\omega}p_i, \tag{2.12}$$

where $c, d \in \mathbb{R}$ are arbitrary constants. We have identified M as the topologically generated "photon" mass of the Abelian gauge field, which is proportional to the prefactor k of the Chern-Simons term. In the absence of the Maxwell term, i.e. when $e^2 \to \infty$, the gauge field becomes infinitely heavy. In the pure Maxwell theory (with $k = 0$), on the other hand, the gauge field remains massless, and plane waves are purely transverse.

In order to quantize the theory, we now impose canonical commutation relations

$$[\Pi_i(x), A_j(y)] = -i\delta_{ij}\delta(x - y), \tag{2.13}$$

which imply the following commutation relations between the electric and magnetic fields

$$\begin{aligned}
&[E_i(x), E_j(y)] = -i\frac{k}{2\pi}\epsilon_{ij}\delta(x - y), \\
&[B(x), B(y)] = 0, \\
&[E_i(x), B(y)] = i\epsilon_{ij}\partial_j\delta(x - y).
\end{aligned} \tag{2.14}$$

The derivative in the third equation is with respect to x (not y). As a consequence of the Chern-Simons term, the two components of the electric field do not commute with each other. It is easy to check that the Hamiltonian

$$H = \int d^2x \left(\frac{e^2}{2}E_i^2 + \frac{1}{2e^2}B^2\right) \tag{2.15}$$

commutes with the infinitesimal generators of local gauge transformations

$$G(x) = \partial_i E_i(x) + \frac{k}{2\pi} B(x), \quad [G(x), H] = 0. \tag{2.16}$$

As usual in a gauge theory, physical states $|\Psi\rangle$ must obey the Gauss law $G(x)|\Psi\rangle = 0$. Upon quantization, the classical plane wave solutions then turn into free "photon" states of mass $M = \frac{ke^2}{2\pi}$.

3 Doubled Continuum Theory with two Abelian Gauge Fields

In the next section, we will regularize Chern-Simons-Maxwell theory on a particular lattice, which will result in a doubling problem, similar to the well-known lattice fermion doubling problem. The continuum limit of that lattice theory is a doubled Chern-Simons-Maxwell theory with two Abelian gauge fields, which we first investigate directly in the continuum.

3.1 Lagrangian and Hamiltonian

We now consider two Abelian gauge fields $A_\mu(x)$ and $\widetilde{A}_\mu(x)$ with the Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4\widetilde{e}^2} \widetilde{F}_{\mu\nu} \widetilde{F}^{\mu\nu} + \frac{k}{4\pi} \epsilon_{\mu\nu\rho} (A^\mu \partial^\nu \widetilde{A}^\rho + \widetilde{A}^\mu \partial^\nu A^\rho). \tag{3.1}$$

Without loss of generality one can assume that the dual charge equals the original charge (i.e. $\widetilde{e} = e$). If this is not the case a priori, one can achieve this by a simple rescaling of the fields. Unlike the theory with a single gauge field, the doubled theory is invariant under time-reversal and parity, because one can treat $\widetilde{A}_\mu(x)$ as a pseudo-vector.

Again, fixing to the temporal gauge $A_0(x) = 0$, $\widetilde{A}_0(x) = 0$, one obtains

$$\mathcal{L} = \frac{1}{2e^2} \dot{A}_i^2 - \frac{1}{2e^2} (\epsilon_{ij} \partial_i A_j)^2 + \frac{1}{2e^2} \dot{\widetilde{A}}_i^2 - \frac{1}{2e^2} (\epsilon_{ij} \partial_i \widetilde{A}_j)^2 + \frac{k}{4\pi} \epsilon_{ij} (\dot{A}_i \widetilde{A}_j + \dot{\widetilde{A}}_i A_j), \tag{3.2}$$

which yields the following canonically conjugate momenta

$$\begin{aligned} \Pi_i(x) &= \frac{\delta \mathcal{L}}{\delta \dot{A}_i(x)} = \frac{1}{e^2} \dot{A}_i(x) + \frac{k}{4\pi} \epsilon_{ij} \widetilde{A}_j(x), \\ \widetilde{\Pi}_i(x) &= \frac{\delta \mathcal{L}}{\delta \dot{\widetilde{A}}_i(x)} = \frac{1}{e^2} \dot{\widetilde{A}}_i(x) + \frac{k}{4\pi} \epsilon_{ij} A_j(x). \end{aligned} \tag{3.3}$$

The classical Hamilton density then takes the form

$$
\begin{aligned}
\mathcal{H} &= \Pi_i \dot{A}_i + \widetilde{\Pi}_i \dot{\widetilde{A}}_i - \mathcal{L} = \frac{1}{2e^2}\dot{A}_i^2 + \frac{1}{2e^2}(\epsilon_{ij}\partial_i A_j)^2 + \frac{1}{2e^2}\dot{\widetilde{A}}_i^2 + \frac{1}{2e^2}(\epsilon_{ij}\partial_i \widetilde{A}_j)^2 \\
&= \frac{e^2}{2}\left(\Pi_i - \frac{k}{4\pi}\epsilon_{ij}\widetilde{A}_j\right)^2 + \frac{1}{2e^2}(\epsilon_{ij}\partial_i A_j)^2 + \frac{e^2}{2}\left(\widetilde{\Pi}_i - \frac{k}{4\pi}\epsilon_{ij}A_j\right)^2 + \frac{1}{2e^2}(\epsilon_{ij}\partial_i \widetilde{A}_j)^2 \\
&= \frac{e^2}{2}E_i^2 + \frac{1}{2e^2}B^2 + \frac{e^2}{2}\widetilde{E}_i^2 + \frac{1}{2e^2}\widetilde{B}^2. \qquad (3.4)
\end{aligned}
$$

In this case, the electric and magnetic fields are given by

$$
\begin{aligned}
E_i(x) &= \Pi_i(x) - \frac{k}{4\pi}\epsilon_{ij}\widetilde{A}_j(x), \quad B(x) = \partial_1 A_2(x) - \partial_2 A_1(x), \\
\widetilde{E}_i(x) &= \widetilde{\Pi}_i(x) - \frac{k}{4\pi}\epsilon_{ij}A_j(x), \quad \widetilde{B}(x) = \partial_1 \widetilde{A}_2(x) - \partial_2 \widetilde{A}_1(x). \qquad (3.5)
\end{aligned}
$$

3.2 Solutions of the Classical Equations of Motion

The classical equations of motion of the doubled theory are

$$
\partial^\mu F_{\mu\nu}(x) + \frac{ke^2}{4\pi}\epsilon_{\nu\rho\sigma}\widetilde{F}^{\rho\sigma}(x) = 0, \quad \partial^\mu \widetilde{F}_{\mu\nu}(x) + \frac{ke^2}{4\pi}\epsilon_{\nu\rho\sigma}F^{\rho\sigma}(x) = 0, \qquad (3.6)
$$

which in components take the form

$$
\begin{aligned}
\dot{E}_i(x) + \frac{1}{e^2}\epsilon_{ij}\partial_j B(x) + \frac{ke^2}{2\pi}\epsilon_{ij}\widetilde{E}_j(x) = 0, &\quad \partial_i E_i(x) + \frac{k}{2\pi}\widetilde{B}(x) = 0, \\
\dot{\widetilde{E}}_i(x) + \frac{1}{e^2}\epsilon_{ij}\partial_j \widetilde{B}(x) + \frac{ke^2}{2\pi}\epsilon_{ij}E_j(x) = 0, &\quad \partial_i \widetilde{E}_i(x) + \frac{k}{2\pi}B(x) = 0. \qquad (3.7)
\end{aligned}
$$

As before, we make the plane wave ansatz

$$
\begin{aligned}
E_i(x) &= C_i \cos(\vec{p}\cdot\vec{x} - \omega t) + D_i \sin(\vec{p}\cdot\vec{x} - \omega t), \\
\widetilde{E}_i(x) &= \widetilde{C}_i \cos(\vec{p}\cdot\vec{x} - \omega t) + \widetilde{D}_i \sin(\vec{p}\cdot\vec{x} - \omega t), \qquad (3.8)
\end{aligned}
$$

which again implies $\omega = \sqrt{M^2 + p^2}$, and $M = \frac{ke^2}{2\pi}$, as well as

$$
\begin{aligned}
C_i &= c\epsilon_{ij}p_j - \widetilde{d}\frac{M}{\omega}p_i, \quad D_i = d\epsilon_{ij}p_j + \widetilde{c}\frac{M}{\omega}p_i, \\
\widetilde{C}_i &= \widetilde{c}\epsilon_{ij}p_j - d\frac{M}{\omega}p_i, \quad \widetilde{D}_i = \widetilde{d}\epsilon_{ij}p_j + c\frac{M}{\omega}p_i. \qquad (3.9)
\end{aligned}
$$

In this case, there are four independent amplitudes $c, d, \widetilde{c}, \widetilde{d} \in \mathbb{R}$, and thus there are two independent "photon" modes, both with the same mass M.

3.3 Canonical Quantization

Canonical quantization now amounts to

$$[\Pi_i(x), A_j(y)] = [\widetilde{\Pi}_i(x), \widetilde{A}_j(y)] = -i\delta_{ij}\delta(x-y),$$
$$[\Pi_i(x), \widetilde{A}_j(y)] = [\widetilde{\Pi}_i(x), A_j(y)] = 0, \tag{3.10}$$

which imply the following commutation relations between the electric and magnetic fields

$$\begin{aligned}
&[E_i(x), \widetilde{E}_j(y)] = [\widetilde{E}_i(x), E_j(y)] = -i\frac{k}{2\pi}\epsilon_{ij}\delta(x-y),\\
&[E_i(x), E_j(y)] = [\widetilde{E}_i(x), \widetilde{E}_j(y)] = 0,\\
&[B(x), B(y)] = [\widetilde{B}(x), \widetilde{B}(y)] = [B(x), \widetilde{B}(y)] = 0,\\
&[E_i(x), B(y)] = [\widetilde{E}_i(x), \widetilde{B}(y)] = i\epsilon_{ij}\partial_j\delta(x-y),\\
&[E_i(x), \widetilde{B}(y)] = [\widetilde{E}_i(x), B(y)] = 0.
\end{aligned} \tag{3.11}$$

The Hamiltonian now takes the form

$$H = \int d^2x \left(\frac{e^2}{2}E_i^2 + \frac{1}{2e^2}B^2 + \frac{e^2}{2}\widetilde{E}_i^2 + \frac{1}{2e^2}\widetilde{B}^2\right). \tag{3.12}$$

It commutes with the infinitesimal generators of two local gauge transformations

$$G(x) = \partial_i E_i(x) + \frac{k}{2\pi}\widetilde{B}(x), \quad \widetilde{G}(x) = \partial_i \widetilde{E}_i(x) + \frac{k}{2\pi}B(x), \quad [G(x), H] = [\widetilde{G}(x), H] = 0. \tag{3.13}$$

The Gauss law now constrains physical states to $G(x)|\Psi\rangle = \widetilde{G}(x)|\Psi\rangle = 0$. The classical plane wave solutions then turn into the states of two free "photons", both of mass $M = \frac{ke^2}{2\pi}$.

4 Non-compact Chern-Simons-Maxwell Theory on the Lattice

In this section, we regularize Chern-Simons-Maxwell theory on a particular spatial lattice. We work in the Hamiltonian formulation and thus leave time continuous. As we will see, the lattice theory suffers from a doubling problem and thus reduces to the doubled Chern-Simons-Maxwell theory in the continuum limit.

4.1 Cross-based Degrees of Freedom

For simplicity, we consider a square lattice. The generalization to other lattice geometries is straightforward but not very illuminating. Most previous lattice constructions of Chern-Simons theories have placed all dynamical gauge degrees of

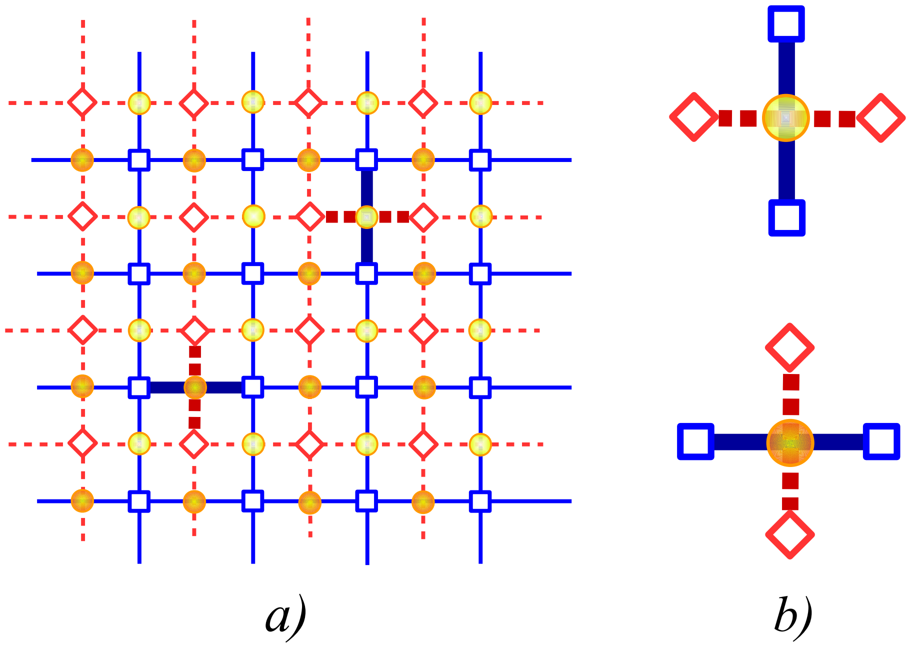

Figure 1: [Color online] *a) Square lattice* Λ *(squares, solid links) together with its dual lattice* $\widetilde{\Lambda}$ *(diamonds, dashed links). The gauge fields* $A_{x,i}$ *and* $\widetilde{A}_{x,i}$ *are associated with two links forming a cross centered at the point* x*. The cross centers* x *do not belong to the lattices* Λ *or* $\widetilde{\Lambda}$*, but form two other sublattices* X *(filled circles) and* $\widetilde{X}$ *(open circles). b) Two types of crosses of link-pairs,* $A_{x,1}, \widetilde{A}_{x,2}$ *and* $\widetilde{A}_{x,1}, A_{x,2}$*, whose cross centers* x *belong to the two distinct sublattices* X *and* $\widetilde{X}$*, respectively.*

freedom on the same lattice. Here, similar to the lattice geometry used in [46], we simultaneously place independent gauge degrees of freedom on the original as well as on the dual lattice. This is natural from a geometric point of view, but it inevitably leads to the doubling of the massive “photon” mode. This is no problem in the present context, because it is our goal to explicitly connect lattice Chern-Simons-Maxwell theory to the toric code, which indeed requires to start from the doubled theory.

In standard lattice gauge theory, the basic gauge field algebra is link-based, i.e. the gauge field variables as well as their canonically conjugate electric field operators are defined on a link. In particular, field operators residing on different links commute with one another. As we have seen, in the doubled Chern-Simons-Maxwell theory the canonically conjugate variable to A_i is $\Pi_i = \frac{1}{e^2}\dot{A}_i + \frac{k}{4\pi}\epsilon_{ij}\widetilde{A}_j$. For the corresponding lattice theory, this implies that the gauge variables associated with different links no longer necessarily commute. As we will see, it is natural to associate the non-commuting gauge field variables with a link and the corresponding dual link. Consequently, the basic gauge algebra of doubled lattice Chern-Simons theory is not based on a single link, but on a pair of links which are dual to each other. The original square lattice together with its dual lattice is illustrated in Fig.1.

Let us consider a link and the corresponding dual link which form a cross centered at a point x. Note that x is neither a site of the original nor of the dual lattice, but just an intersection point of the link and its dual link on a sublattice X or $\widetilde{X}$ (cf. Fig.1). We denote the non-compact vector potential on the original link by $A_{x,i}$. The vector potential on the corresponding dual link is then given by $\epsilon_{ij}\widetilde{A}_{x,j}$. It should be noted that for a given cross, the direction i of the original link variable $A_{x,i}$ and the orthogonal direction j of the dual link variable $\widetilde{A}_{x,j}$ are determined by the location of the intersection point x on sublattice X or $\widetilde{X}$.

4.2 Lagrangian and Hamiltonian

In the continuum the integrated Lagrange density (i.e. the Lagrange function) is given by

$$L = \int d^2x \left[\frac{1}{2e^2}\dot{A}_i^2 - \frac{1}{2e^2}B^2 + \frac{1}{2e^2}\dot{\widetilde{A}}_i^2 - \frac{1}{2e^2}\widetilde{B}^2 + \frac{k}{4\pi}\epsilon_{ij}(\dot{A}_i\widetilde{A}_j + \dot{\widetilde{A}}_i A_j)\right]. \tag{4.1}$$

Correspondingly, on the lattice we obtain

$$\begin{aligned} L &= \sum_{x\in X} a^2\left[\frac{1}{2e^2}\dot{A}_{x,1}^2 + \frac{1}{2e^2}\dot{\widetilde{A}}_{x,2}^2 + \frac{k}{4\pi}(\dot{A}_{x,1}\widetilde{A}_{x,2} - \dot{\widetilde{A}}_{x,2}A_{x,1})\right] \\ &+ \sum_{x\in\widetilde{X}} a^2\left[\frac{1}{2e^2}\dot{A}_{x,2}^2 + \frac{1}{2e^2}\dot{\widetilde{A}}_{x,1}^2 + \frac{k}{4\pi}(\dot{\widetilde{A}}_{x,1}A_{x,2} - \dot{A}_{x,2}\widetilde{A}_{x,1})\right] \\ &- \sum_{x\in\widetilde{\Lambda}} a^2\frac{1}{2e^2}B_x^2 - \sum_{x\in\Lambda} a^2\frac{1}{2e^2}\widetilde{B}_x^2. \end{aligned} \tag{4.2}$$

In the sums over the sublattices X, $\widetilde{X}$, Λ, $\widetilde{\Lambda}$ the factors a^2 (where a is the spacing of the original or the dual lattice) arises as the area of an elementary plaquette. On the lattice, the magnetic fields are defined as

$$\begin{aligned} B_x &= \frac{1}{a}\left(A_{x-\frac{a}{2}\hat{2},1} + A_{x+\frac{a}{2}\hat{1},2} - A_{x+\frac{a}{2}\hat{2},1} - A_{x-\frac{a}{2}\hat{1},2}\right), \\ \widetilde{B}_x &= \frac{1}{a}\left(\widetilde{A}_{x-\frac{a}{2}\hat{2},1} + \widetilde{A}_{x+\frac{a}{2}\hat{1},2} - \widetilde{A}_{x+\frac{a}{2}\hat{2},1} - \widetilde{A}_{x-\frac{a}{2}\hat{1},2}\right), \end{aligned} \tag{4.3}$$

with $\hat{i}$ being the unit-vector in the i-direction. Note that the magnetic field B_x, which is built around a plaquette of the original lattice, is indexed by the dual site $x \in \widetilde{\Lambda}$ at the center of this plaquette. The magnetic fields are invariant against lattice gauge transformations

$$A'_{x,i} = A_{x,i} - \frac{1}{a}\left(\chi_{x+\frac{a}{2}\hat{i}} - \chi_{x-\frac{a}{2}\hat{i}}\right), \quad \widetilde{A}'_{x,i} = \widetilde{A}_{x,i} - \frac{1}{a}\left(\widetilde{\chi}_{x+\frac{a}{2}\hat{i}} - \widetilde{\chi}_{x-\frac{a}{2}\hat{i}}\right). \tag{4.4}$$

On the lattice, the canonically conjugate momenta are given by

$$\Pi_{x,i} = \frac{\partial L}{\partial \dot{A}_{x,i}} = a^2 \left(\frac{1}{e^2} \dot{A}_{x,i} + \frac{k}{4\pi} \epsilon_{ij} \widetilde{A}_{x,j} \right),$$
$$\widetilde{\Pi}_{x,i} = \frac{\partial L}{\partial \dot{\widetilde{A}}_{x,i}} = a^2 \left(\frac{1}{e^2} \dot{\widetilde{A}}_{x,i} + \frac{k}{4\pi} \epsilon_{ij} A_{x,j} \right). \quad (4.5)$$

Note that on the lattice $\Pi_{x,i}$ has a different dimension than $\Pi_i(x)$ in the continuum. The corresponding classical Hamilton function then takes the form

$$\begin{aligned} H &= \sum_{x \in X} (\Pi_{x,1} \dot{A}_{x,1} + \widetilde{\Pi}_{x,2} \dot{\widetilde{A}}_{x,2}) + \sum_{x \in \widetilde{X}} (\Pi_{x,2} \dot{A}_{x,2} + \widetilde{\Pi}_{x,1} \dot{\widetilde{A}}_{x,1}) - L \\ &= \frac{e^2}{2} \sum_{x \in X} a^2 (E_{x,1}^2 + \widetilde{E}_{x,2}^2) + \frac{e^2}{2} \sum_{x \in \widetilde{X}} a^2 (E_{x,2}^2 + \widetilde{E}_{x,1}^2) \\ &+ \frac{1}{2e^2} \sum_{x \in \widetilde{\Lambda}} a^2 B_x^2 + \frac{1}{2e^2} \sum_{x \in \Lambda} a^2 \widetilde{B}_x^2. \end{aligned} \quad (4.6)$$

In this case, the electric fields are given by

$$E_{x,i} = \frac{1}{a^2} \Pi_{x,i} - \frac{k}{4\pi} \epsilon_{ij} \widetilde{A}_{x,j}, \quad \widetilde{E}_{x,i} = \frac{1}{a^2} \widetilde{\Pi}_{x,i} - \frac{k}{4\pi} \epsilon_{ij} A_{x,j}. \quad (4.7)$$

4.3 Solutions of the Classical Equations of Motion

Before we quantize the theory, let us first consider the classical equations of motion on the lattice, which in components take the form

$$\begin{aligned} &\sum_i \frac{1}{a} (E_{x+\frac{a}{2}\hat{i},i} - E_{x-\frac{a}{2}\hat{i},i}) + \frac{k}{2\pi} \widetilde{B}_x = 0, \\ &\dot{E}_{x,i} + \frac{1}{e^2} \epsilon_{ij} \frac{1}{a} (B_{x+\frac{a}{2}\hat{j}} - B_{x-\frac{a}{2}\hat{j}}) + \frac{ke^2}{2\pi} \epsilon_{ij} \widetilde{E}_{x,j} = 0, \\ &\sum_i \frac{1}{a} (\widetilde{E}_{x+\frac{a}{2}\hat{i},i} - \widetilde{E}_{x-\frac{a}{2}\hat{i},i}) + \frac{k}{2\pi} B_x = 0, \\ &\dot{\widetilde{E}}_{x,i} + \frac{1}{e^2} \epsilon_{ij} \frac{1}{a} (\widetilde{B}_{x+\frac{a}{2}\hat{j}} - \widetilde{B}_{x-\frac{a}{2}\hat{j}}) + \frac{ke^2}{2\pi} \epsilon_{ij} E_{x,j} = 0. \end{aligned} \quad (4.8)$$

The first equation corresponds to the lattice Gauss law, which is illustrated in Fig.2a. The geometry underlying the second equation is depicted in Fig.2b. As in the continuum, we make the plane wave ansatz

$$\begin{aligned} E_{x,i} &= C_i \cos(\vec{p} \cdot \vec{x} - \omega t) + D_i \sin(\vec{p} \cdot \vec{x} - \omega t), \\ \widetilde{E}_{x,i} &= \widetilde{C}_i \cos(\vec{p} \cdot \vec{x} - \omega t) + \widetilde{D}_i \sin(\vec{p} \cdot \vec{x} - \omega t). \end{aligned} \quad (4.9)$$

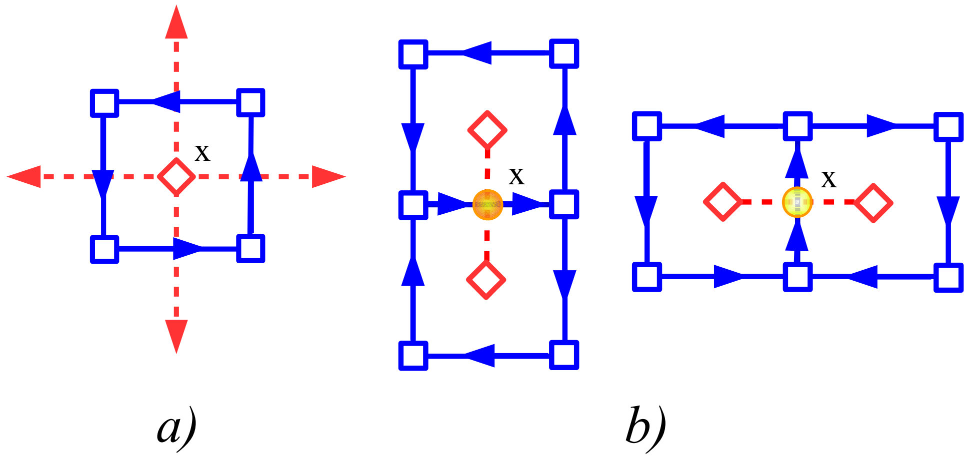

Figure 2: [Color online] *Geometry of the lattice equations of motion: a) The Gauss law arises from the divergence of the electric field, i.e. from the electric fluxes flowing out of a lattice point x, and from the magnetic field associated with a plaquette formed with the dual links. b) The other two equations of motion relate the time-derivative of an electric field $\dot{E}_{x,i}$ on a link centered at x to the dual electric field $\epsilon_{ij}\widetilde{E}_{x,j}$ as well as to the difference of magnetic fields residing on two nearest-neighbor plaquettes.*

Inserting this in the lattice equations of motion, we obtain the dispersion relation

$$\omega = \sqrt{M^2 + \hat{p}^2}, \quad M = \frac{ke^2}{2\pi}, \quad \hat{p}_i = \frac{2}{a} \sin \frac{p_i a}{2}. \tag{4.10}$$

Note that the mass of the gauge field is the same as in the continuum, while the energy-momentum dispersion relation is distorted by lattice artifacts for lattice momenta p close to the edge of the Brillouin zone. In analogy to the continuum result, we obtain

$$\begin{aligned} C_i &= c\epsilon_{ij}\hat{p}_j - \widetilde{d}\frac{M}{\omega}\hat{p}_i, \quad D_i = d\epsilon_{ij}\hat{p}_j + \widetilde{c}\frac{M}{\omega}\hat{p}_i, \\ \widetilde{C}_i &= \widetilde{c}\epsilon_{ij}\hat{p}_j - d\frac{M}{\omega}\hat{p}_i, \quad \widetilde{D}_i = \widetilde{d}\epsilon_{ij}\hat{p}_j + c\frac{M}{\omega}\hat{p}_i. \end{aligned} \tag{4.11}$$

As in the continuum, there are four independent amplitudes $c, d, \widetilde{c}, \widetilde{d} \in \mathbb{R}$, and thus the massive "photon" mode is again doubled.

4.4 Canonical Quantization

Upon canonical quantization we now obtain

$$\begin{aligned} [\Pi_{x,i}, A_{y,j}] &= [\widetilde{\Pi}_{x,i}, \widetilde{A}_{y,j}] = -i\delta_{ij}\delta_{xy}, \\ [\Pi_{x,i}, \widetilde{A}_{y,j}] &= [\widetilde{\Pi}_{x,i}, A_{y,j}] = 0. \end{aligned} \tag{4.12}$$

The resulting commutation relations for the vector potential and the electric field are

$$\begin{aligned}
&[A_{x,i}, A_{y,j}] = [\widetilde{A}_{x,i}, \widetilde{A}_{y,j}] = [A_{x,i}, \widetilde{A}_{y,j}] = 0, \\
&[E_{x,i}, A_{y,j}] = [\widetilde{E}_{x,i}, \widetilde{A}_{y,j}] = -i\delta_{ij}\frac{1}{a^2}\delta_{xy}, \\
&[E_{x,i}, \widetilde{A}_{y,j}] = [\widetilde{E}_{x,i}, A_{y,j}] = 0, \\
&[E_{x,i}, \widetilde{E}_{y,j}] = [\widetilde{E}_{x,i}, E_{y,j}] = -i\frac{k}{2\pi}\epsilon_{ij}\frac{1}{a^2}\delta_{xy}, \\
&[E_{x,i}, E_{y,j}] = [\widetilde{E}_{x,i}, \widetilde{E}_{y,j}] = 0.
\end{aligned} \tag{4.13}$$

In the continuum limit $a \to 0$, $\frac{1}{a^2}\delta_{xy}$ approaches $\delta(x-y)$. It is interesting to note that the commutation relations between $A_{x,i}$ and $\epsilon_{ij}\widetilde{A}_{x,j}$ with $E_{x,i}$ and $\epsilon_{ij}\widetilde{E}_{x,j}$, which reside on the cross formed by a link and its dual link, correspond to those of a charged particle moving in a 2-d plane in a constant magnetic field $\frac{k}{2\pi}$. In this "mechanical" analog, $A_{x,i}$ and $\epsilon_{ij}\widetilde{A}_{x,j}$ play the role of the particle's x- and y-coordinate, while $E_{x,i}$ and $\epsilon_{ij}\widetilde{E}_{x,j}$ represent the corresponding momenta. Like for a charged particle in a magnetic field, the commutator of the momenta is non-zero and given by the abstract "magnetic" field $\frac{k}{2\pi}$. In the next section, we will compactify the gauge field from $\mathbb{R}$ to $U(1)$, which will naturally lead to a quantization of the abstract "magnetic" field in integer units $k \in \mathbb{Z}$. For the moment, however, k is not quantized.

The commutation relations between electric and magnetic fields now take the form

$$\begin{aligned}
&[E_{x,i}, B_y] = [\widetilde{E}_{x,i}, \widetilde{B}_y] = i\epsilon_{ij}\frac{1}{a^3}(\delta_{x,y+\frac{a}{2}\hat{j}} - \delta_{x,y-\frac{a}{2}\hat{j}}), \\
&[E_{x,i}, \widetilde{B}_y] = [\widetilde{E}_{x,i}, B_y] = 0, \\
&[B_x, B_y] = [\widetilde{B}_x, \widetilde{B}_y] = [B_x, \widetilde{B}_y] = 0.
\end{aligned} \tag{4.14}$$

The Hamiltonian of eq.(4.6) commutes with the infinitesimal generators of local gauge transformations, both on the original and on the dual lattice

$$\begin{aligned}
\frac{G_x}{a^2} &= \sum_i \frac{1}{a}(E_{x+\frac{a}{2}\hat{i},i} - E_{x-\frac{a}{2}\hat{i},i}) + \frac{k}{2\pi}\widetilde{B}_x, \quad [G_x, H] = 0, \quad x \in \Lambda, \\
\frac{\widetilde{G}_x}{a^2} &= \sum_i \frac{1}{a}(\widetilde{E}_{x+\frac{a}{2}\hat{i},i} - \widetilde{E}_{x-\frac{a}{2}\hat{i},i}) + \frac{k}{2\pi}B_x, \quad [\widetilde{G}_x, H] = 0, \quad x \in \widetilde{\Lambda}.
\end{aligned} \tag{4.15}$$

Note that G_x has a different dimension than $G(x)$ in the continuum. In general, generators of gauge transformations at different sites of the same sublattice commute with each other, i.e. $[G_x, G_y] = 0$, $[\widetilde{G}_x, \widetilde{G}_y] = 0$. Let us also compute the commutation relations of the generator of a gauge transformation on the original

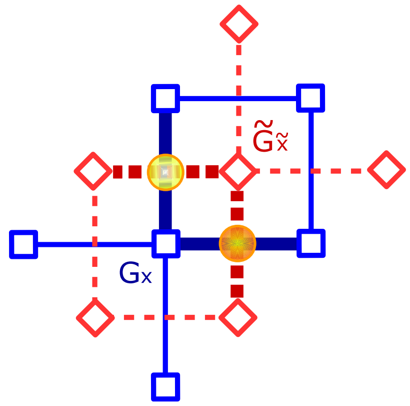

Figure 3: [Color online] *Consistency of the Gauss laws on the original and on the dual lattice. The gauge generators* G_x *and* $\widetilde{G}_{\widetilde{x}}$ *(with* $\widetilde{x} = x + \frac{a}{2}\hat{1} + \frac{a}{2}\hat{2}$*) commute, i.e.* $[G_x, \widetilde{G}_{\widetilde{x}}] = 0$*. This is non-trivial, because, e.g., the electric fields* $E_{x+\frac{a}{2}\hat{1},1}$ *and* $\widetilde{E}_{\widetilde{x}-\frac{a}{2}\hat{2},2}$ *that contribute to* G_x *and* $\widetilde{G}_{\widetilde{x}}$ *do not commute since they belong to the same cross formed by a link and its dual link.*

lattice and a neighboring site on the dual lattice (cf. Fig.3)

$$\begin{aligned}
[G_x, \widetilde{G}_{x+\frac{a}{2}\hat{1}+\frac{a}{2}\hat{2}}] &= -a^2[E_{x+\frac{a}{2}\hat{1},1}, \widetilde{E}_{x+\frac{a}{2}\hat{1},2}] - a^2[E_{x+\frac{a}{2}\hat{2},2}, \widetilde{E}_{x+\frac{a}{2}\hat{2},1}] \\
&+ \frac{ka^3}{2\pi}[E_{x+\frac{a}{2}\hat{1},1}, B_{x+\frac{a}{2}\hat{1}+\frac{a}{2}\hat{2}}] + \frac{ka^3}{2\pi}[E_{x+\frac{a}{2}\hat{2},2}, B_{x+\frac{a}{2}\hat{1}+\frac{a}{2}\hat{2}}] \\
&+ \frac{ka^3}{2\pi}[\widetilde{E}_{x+\frac{a}{2}\hat{2},1}, \widetilde{B}_x] + \frac{ka^3}{2\pi}[\widetilde{E}_{x+\frac{a}{2}\hat{1},2}, \widetilde{B}_x] = 0. \qquad (4.16)
\end{aligned}$$

Hence, generators of gauge transformations at different sites always commute.

The vacuum state $|0\rangle$ is gauge invariant and satisfies the Gauss law $G_x|0\rangle = \widetilde{G}_x|0\rangle = 0$. Upon quantization, the classical plane wave states turn into the states of two "photons", each of mass $M = \frac{ke^2}{2\pi}$, with the lattice dispersion relation $E(p) = \sqrt{M^2 + \hat{p}^2}$, $\hat{p}_i = \frac{2}{a}\sin\frac{p_i a}{2}$. A state $|Q, \widetilde{Q}\rangle$ that contains static charges $Q_x \in \mathbb{Z}$ (with $x \in \Lambda$) and static dual charges $\widetilde{Q}_x \in \mathbb{Z}$ (with $x \in \widetilde{\Lambda}$) satisfies

$$G_x|Q, \widetilde{Q}\rangle = Q_x|Q, \widetilde{Q}\rangle,\ x \in \Lambda, \quad \widetilde{G}_x|Q, \widetilde{Q}\rangle = \widetilde{Q}_x|Q, \widetilde{Q}\rangle,\ x \in \widetilde{\Lambda}. \qquad (4.17)$$

4.5 Mutual Statistics of Original and Dual Charges

Let us consider Wilson's parallel transporter $U_{x,i} = \exp(iaA_{x,i})$, which moves a charge from $x - \frac{a}{2}\hat{i}$ to $x + \frac{a}{2}\hat{i}$. Under the unitary transformation

$$V = \prod_{x \in \Lambda} V_x = \prod_{x \in \Lambda} \exp(i\chi_x G_x), \qquad (4.18)$$

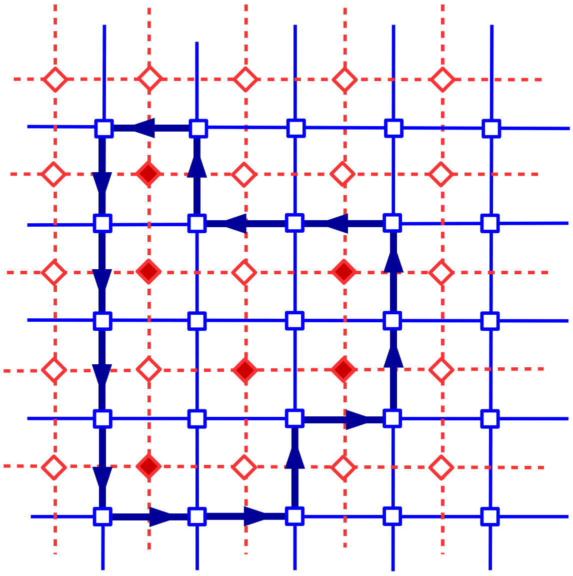

Figure 4: [Color online] *A Wilson loop $W_{\mathcal{C}}$ on the original lattice encircles a set of dual charges $\widetilde{Q}_x$ (filled diamonds). Up to exponentially suppressed corrections due to the finite "photon" mass, the vacuum expectation value of the Wilson loop depends only on the total encircled dual charge, not on further details of the closed loop $\mathcal{C}$.*

which implements a general gauge transformation in Hilbert space, the parallel transporter transforms as

$$U'_{x,i} = V U_{x,i} V^\dagger = \exp(i\chi_{x-\frac{a}{2}\hat{i}}) U_{x,i} \exp(-i\chi_{x+\frac{a}{2}\hat{i}}), \quad x \in X, \widetilde{X}. \tag{4.19}$$

The corresponding commutation relation takes the form

$$[G_y, U_{x,i}] = \frac{1}{a}\sum_j [\Pi_{y+\frac{a}{2}\hat{j},j} - \Pi_{y-\frac{a}{2}\hat{j},j}, U_{x,i}] = \left(\delta_{y+\frac{a}{2}\hat{i},x} - \delta_{y-\frac{a}{2}\hat{i},x}\right) U_{x,i}. \tag{4.20}$$

Let us now act with $U_{x,i}$ on the charged state $|Q,\widetilde{Q}\rangle$, with external charges Q_x at $x \in \Lambda$ and $\widetilde{Q}_x$ at $x \in \widetilde{\Lambda}$ (cf. eq.(4.17)), in order to investigate its effect on the charges

$$\begin{aligned} G_y|Q',\widetilde{Q}\rangle &= G_y U_{x,i}|Q,\widetilde{Q}\rangle = U_{x,i} G_y |Q,\widetilde{Q}\rangle + \left(\delta_{y+\frac{a}{2}\hat{i},x} - \delta_{y-\frac{a}{2}\hat{i},x}\right) U_{x,i}|Q,\widetilde{Q}\rangle \\ &= \left(Q_y + \delta_{y+\frac{a}{2}\hat{i},x} - \delta_{y-\frac{a}{2}\hat{i},x}\right)|Q',\widetilde{Q}\rangle. \end{aligned} \tag{4.21}$$

As expected $U_{x,i}$ acts as a charge transport operator which moves a charge from $x - \frac{a}{2}\hat{i}$ to $x + \frac{a}{2}\hat{i}$, i.e. the state $|Q',\widetilde{Q}\rangle = U_{x,i}|Q,\widetilde{Q}\rangle$ has charges

$$Q'_y = Q_y + \delta_{y+\frac{a}{2}\hat{i},x} - \delta_{y-\frac{a}{2}\hat{i},x}. \tag{4.22}$$

Let us now transport a charge around the closed loop $\mathcal{C}$ (on the original lattice), so that it returns to its initial position (cf. Fig.4). The wave function then turns into

$$W_{\mathcal{C}}|Q,\widetilde{Q}\rangle = \prod_{(x,i)\in\mathcal{C}} U_{x,i}|Q,\widetilde{Q}\rangle = \exp(i\Phi_{\mathcal{C}})|Q,\widetilde{Q}\rangle, \tag{4.23}$$

i.e., it picks up a phase

$$\Phi_{\mathcal{C}} = \sum_{x\in S\subset\widetilde{\Lambda}} a^2 B_x = \sum_{x\in S\subset\widetilde{\Lambda}} \frac{2\pi}{k}\widetilde{G}_x - \sum_{\mathcal{C}} \frac{2\pi a}{k}\epsilon_{ij}\widetilde{E}_{x,j}, \tag{4.24}$$

which is given by the magnetic flux through the surface S that is bounded by $\mathcal{C} = \partial S$. Here x denotes a point on the dual lattice at the center of a plaquette on the original lattice which belongs to the surface S. Eq.(4.24) results from Stoke's theorem on the lattice as well as from eq.(4.15). The first term on the right-hand side of eq.(4.24) counts the dual charges $\widetilde{Q}_S = \sum_{x\in S\subset\widetilde{\Lambda}} \widetilde{G}_x$ encircled by the loop $\mathcal{C}$. In a pure Chern-Simons theory (without a Maxwell term) this would be the only contribution. In that case, a charge that is transported around a closed loop $\mathcal{C}$ on the original lattice picks up a topological phase $\exp(2\pi i\widetilde{Q}_S/k)$. In other words, charges on the original and on the dual lattice have mutual anyonic statistics, with the statistics angle $\frac{2\pi}{k}$. The topological phase is an adiabatic Berry phase. However, in contrast to generic Berry phases, the topological phase $\Phi_{\mathcal{C}}$ is even insensitive to the shape of the curve $\mathcal{C}$, as long as it encircles the same total charge $\widetilde{Q}_S$ on the dual lattice. Obviously, corresponding rules about charge transport also apply to the dual lattice, with $U_{x,i}$ being replaced by $\widetilde{U}_{x,i} = \exp(ia\widetilde{A}_{x,i})$. In the Chern-Simons-Maxwell theory that we investigate here, a second term arises on the right-hand side of eq.(4.24). This term is due to the "photon" cloud surrounding the dual charges. Since the "photon" is massive, the effect of this term is exponentially suppressed at large distances [14]. Still, it slightly affects the topological nature of the Berry phase and causes a certain sensitivity to the precise location of the curve $\mathcal{C}$. The central result of mutual anyonic statistics of original and dual charges with statistics angle $\frac{2\pi}{k}$ remains valid, as long as the charges are moved around each other at distances much larger than the size of their massive "photon" clouds.

5 Compact Chern-Simons-Maxwell Theory on the Lattice

In the previous section, we have considered a non-compact lattice theory with gauge group $\mathbb{R}$. As we have seen in eq.(4.13), the local degrees of freedom on a cross formed by a link and its dual link then have a "mechanical" analog, namely a charged "particle" moving in the 2-d plane $\mathbb{R}^2$ in the background of an abstract "magnetic" field $\frac{k}{2\pi}$. Now we will compactify the gauge field and thus work with the gauge group $U(1)$. The "mechanical" analog then corresponds to motion restricted to the torus $U(1)^2$. This leads to the quantization of the level k in integer units, i.e. $k \in \mathbb{Z}$. As we will see, the electric field part of the Hamiltonian has a 2-parameter family of self-adjoint extensions. Interestingly, fixing the two self-adjoint extension parameters explicitly breaks the $U(1)$ gauge symmetry down to $\mathbb{Z}(k)$.

5.1 Commutation Relations of the Compact Theory

In the compact theory, $A_{x,i}$ and $\widetilde{A}_{x,i}$ turn into phases of the parallel transporters

$$U_{x,i} = \exp(iaA_{x,i}) = \exp(i\varphi_{x,i}) \in U(1), \quad \widetilde{U}_{x,i} = \exp(ia\widetilde{A}_{x,i}) = \exp(i\widetilde{\varphi}_{x,i}) \in U(1), \tag{5.1}$$

which now are the truly fundamental degrees of freedom, and

$$aE_{x,i} = -i\partial_{\varphi_{x,i}} - \frac{k}{4\pi}\epsilon_{ij}\widetilde{\varphi}_{x,j}, \quad a\widetilde{E}_{x,i} = -i\partial_{\widetilde{\varphi}_{x,i}} - \frac{k}{4\pi}\epsilon_{ij}\varphi_{x,j}. \tag{5.2}$$

Based on eq.(4.13), the local commutation relations now take the form

$$\begin{aligned}
&[U_{x,i}, U_{y,j}] = [\widetilde{U}_{x,i}, \widetilde{U}_{y,j}] = [U_{x,i}, \widetilde{U}_{y,j}] = 0, \\
&[E_{x,i}, U_{y,j}] = \frac{1}{a}\delta_{ij}\delta_{xy}U_{x,i}, \quad [\widetilde{E}_{x,i}, \widetilde{U}_{y,j}] = \frac{1}{a}\delta_{ij}\delta_{xy}\widetilde{U}_{x,i}, \\
&[E_{x,i}, \widetilde{U}_{y,j}] = [\widetilde{E}_{x,i}, U_{y,j}] = 0, \\
&[E_{x,i}, \widetilde{E}_{y,j}] = [\widetilde{E}_{x,i}, E_{y,j}] = -i\frac{k}{2\pi a^2}\epsilon_{ij}\delta_{xy}, \\
&[E_{x,i}, E_{y,j}] = [\widetilde{E}_{x,i}, \widetilde{E}_{y,j}] = 0.
\end{aligned} \tag{5.3}$$

Since, unlike $A_{x,i}$ and $\widetilde{A}_{x,i}$, $\varphi_{x,i}$ and $\widetilde{\varphi}_{x,i}$ are angles, they do not represent self-adjoint quantum mechanical operators.

5.2 Self-adjoint Extensions of the Electric Contribution to the Hamiltonian of a Single Cross

Let us consider a single cross, formed by a link and its dual link. For concreteness, we consider a cross $x \in X$, i.e. the link variable φ is associated with the 1-direction, while $\widetilde{\varphi}$ is associated with the 2-direction. The situation for $x \in \widetilde{X}$ is completely analogous. In order to keep the notation simple, in this subsection we suppress the link indices (x,i), which are uniquely determined for a specific cross $x \in X$. The "mechanical" analog of the problem then corresponds to a charged "particle" with "spatial" coordinates $(\varphi, \widetilde{\varphi})$ moving in the 2-dimensional group space subject to the cross-based Hamiltonian

$$H_+ = \frac{e^2a^2}{2}(E_{x,1}^2 + \widetilde{E}_{x,2}^2) = \frac{e^2a^2}{2}(E^2 + \widetilde{E}^2). \tag{5.4}$$

It should be noted that, after compactification, a dual charge $\widetilde{e}$ can no longer be turned into e by a field redefinition. In order to maintain the symmetry between the original and the dual lattice, we simply put $\widetilde{e} = e$. The electric field operators are now given by

$$aE = -i\partial_\varphi + a(\varphi, \widetilde{\varphi}), \quad a\widetilde{E} = -i\partial_{\widetilde{\varphi}} + \widetilde{a}(\varphi, \widetilde{\varphi}). \tag{5.5}$$

Here $(a, \widetilde{a})$ is an abstract vector potential on the group space torus $U(1)^2$ (parametrized by $(\varphi, \widetilde{\varphi})$) with

$$a(\varphi, \widetilde{\varphi}) = -\frac{k}{4\pi}\widetilde{\varphi}, \quad \widetilde{a}(\varphi, \widetilde{\varphi}) = \frac{k}{4\pi}\varphi, \tag{5.6}$$

which gives rise to the abstract "magnetic" field

$$b(\varphi, \widetilde{\varphi}) = \partial_\varphi \widetilde{a}(\varphi, \widetilde{\varphi}) - \partial_{\widetilde{\varphi}} a(\varphi, \widetilde{\varphi}) = \frac{k}{2\pi}. \tag{5.7}$$

The level k thus determines the value of the constant abstract "magnetic" field $b(\varphi, \widetilde{\varphi})$ (which should not be confused with the actual magnetic fields B_x and $\widetilde{B}_x$).

Since the electric field part of the Hamiltonian H_+ contains E^2 and $\widetilde{E}^2$, these operators must be self-adjoint. For mathematical details related to the theory of self-adjoint extensions we refer to [54, 55]. The problem of finding the self-adjoint extensions of H_+ has been discussed in [50] for the "mechanical" analog of a charged particle moving on a torus in a constant magnetic field. The same mathematical solution applies here. The self-adjoint extension parameters enter the boundary conditions for the wave function $\Psi(\varphi, \widetilde{\varphi})$ on the group space torus $U(1)^2$. This wave function will extend to the wave functional of the lattice field theory, once we combine all crosses to form the entire lattice. For the moment, we continue to consider a single cross in isolation.

While the abstract "magnetic" field b is constant, and thus obviously periodic over the group space torus, the abstract vector potential $(a, \widetilde{a})$ obeys the boundary conditions

$$\begin{aligned}
a(\varphi + 2\pi, \widetilde{\varphi}) &= -\frac{k}{4\pi}\widetilde{\varphi} = a(\varphi, \widetilde{\varphi}), \\
a(\varphi, \widetilde{\varphi} + 2\pi) &= -\frac{k}{4\pi}\widetilde{\varphi} - \frac{k}{2} = a(\varphi, \widetilde{\varphi}) - \frac{k}{2}, \\
\widetilde{a}(\varphi + 2\pi, \widetilde{\varphi}) &= \frac{k}{4\pi}\varphi + \frac{k}{2} = \widetilde{a}(\varphi, \widetilde{\varphi}) + \frac{k}{2}, \\
\widetilde{a}(\varphi, \widetilde{\varphi} + 2\pi) &= \frac{k}{4\pi}\varphi = \widetilde{a}(\varphi, \widetilde{\varphi}).
\end{aligned} \tag{5.8}$$

Let us introduce the transition functions

$$\alpha(\widetilde{\varphi}) = -\frac{k}{2}\widetilde{\varphi} + \widetilde{\theta}, \quad \widetilde{\alpha}(\varphi) = \frac{k}{2}\varphi + \theta, \tag{5.9}$$

where $\theta, \widetilde{\theta} \in [0, 2\pi[$ will turn out to be two self-adjoint extension parameters. We can then express the boundary conditions of eq.(5.8) as twisted periodic boundary conditions over the group space torus $U(1)^2$

$$\begin{aligned}
a(\varphi + 2\pi, \widetilde{\varphi}) &= a(\varphi, \widetilde{\varphi}) - \partial_\varphi \alpha(\widetilde{\varphi}), \\
a(\varphi, \widetilde{\varphi} + 2\pi) &= a(\varphi, \widetilde{\varphi}) - \partial_\varphi \widetilde{\alpha}(\varphi), \\
\widetilde{a}(\varphi + 2\pi, \widetilde{\varphi}) &= \widetilde{a}(\varphi, \widetilde{\varphi}) - \partial_{\widetilde{\varphi}} \alpha(\widetilde{\varphi}), \\
\widetilde{a}(\varphi, \widetilde{\varphi} + 2\pi) &= \widetilde{a}(\varphi, \widetilde{\varphi}) - \partial_{\widetilde{\varphi}} \widetilde{\alpha}(\varphi).
\end{aligned} \tag{5.10}$$

As we see, α and $\widetilde{\alpha}$ play the role of abstract gauge transformations (not to be confused with the original lattice gauge symmetry). In order to respect the abstract gauge structure, the wave function Ψ must obey consistent twisted periodic boundary conditions, i.e.

$$\begin{aligned}
\Psi(\varphi + 2\pi, \widetilde{\varphi}) &= \exp(i\alpha(\widetilde{\varphi}))\Psi(\varphi, \widetilde{\varphi}) = \exp\left(-i\frac{k}{2}\widetilde{\varphi} + i\widetilde{\theta}\right)\Psi(\varphi, \widetilde{\varphi}), \\
\Psi(\varphi, \widetilde{\varphi} + 2\pi) &= \exp(i\widetilde{\alpha}(\varphi))\Psi(\varphi, \widetilde{\varphi}) = \exp\left(i\frac{k}{2}\varphi + i\theta\right)\Psi(\varphi, \widetilde{\varphi}).
\end{aligned} \tag{5.11}$$

Note that this equation holds for a cross $x \in X$, while for $x \in \widetilde{X}$ the θ and $\widetilde{\theta}$ terms change sign. Now we see explicitly, that θ and $\widetilde{\theta}$ enter the boundary condition of the wave function, which is typical for self-adjoint extension parameters. The above relations immediately imply

$$\begin{aligned}
\Psi(\varphi + 2\pi, \widetilde{\varphi} + 2\pi) &= \exp\left(-i\frac{k}{2}(\widetilde{\varphi} + 2\pi) + i\widetilde{\theta}\right)\Psi(\varphi, \widetilde{\varphi} + 2\pi) \\
&= \exp\left(-i\frac{k}{2}(\widetilde{\varphi} + 2\pi) + i\widetilde{\theta} + i\frac{k}{2}\varphi + i\theta\right)\Psi(\varphi, \widetilde{\varphi}), \\
\Psi(\varphi + 2\pi, \widetilde{\varphi} + 2\pi) &= \exp\left(i\frac{k}{2}(\varphi + 2\pi) + i\theta\right)\Psi(\varphi + 2\pi, \widetilde{\varphi}) \\
&= \exp\left(i\frac{k}{2}(\varphi + 2\pi) + i\theta - i\frac{k}{2}\widetilde{\varphi} + i\widetilde{\theta}\right)\Psi(\varphi, \widetilde{\varphi}).
\end{aligned} \tag{5.12}$$

Consistency of the two expressions requires $\exp(2\pi i k) = 1$, which leads to the quantization of the level $k \in \mathbb{Z}$. This quantization condition implies that the abstract "magnetic" flux through the group space torus $U(1)^2$ is quantized as well

$$\int_0^{2\pi} d\varphi \int_0^{2\pi} d\widetilde{\varphi}\; b(\varphi, \widetilde{\varphi}) = 2\pi k. \tag{5.13}$$

The quantization of the level k thus manifests itself as a Dirac quantization condition of the abstract "magnetic" flux that threads the group space torus.

Besides the field strength b, the abstract gauge field $(a, \widetilde{a})$ on the group space torus $U(1)^2$ has two additional gauge invariant quantities — the Polyakov loops that arise due to the non-trivial holonomies of the torus

$$\phi(\widetilde{\varphi}) = \int_0^{2\pi} d\varphi \ a(\varphi, \widetilde{\varphi}) + \alpha(\widetilde{\varphi}) = -k\widetilde{\varphi} + \widetilde{\theta},$$
$$\widetilde{\phi}(\varphi) = \int_0^{2\pi} d\widetilde{\varphi} \ \widetilde{a}(\varphi, \widetilde{\varphi}) + \widetilde{\alpha}(\varphi) = k\varphi + \theta. \tag{5.14}$$

Unlike the classical theory, the quantum theory is sensitive to the complex Aharonov-Bohm phases defined by the Polyakov loops

$$\exp(i\phi(\widetilde{\varphi})) = \exp(-ik\widetilde{\varphi} + i\widetilde{\theta}), \quad \exp(i\widetilde{\phi}(\varphi)) = \exp(ik\varphi + i\theta). \tag{5.15}$$

Remarkably, the Polyakov loops explicitly break the continuous $U(1)^2$ translation invariance of the torus down to the discrete translation group $\mathbb{Z}(k)^2$, since the corresponding Aharonov-Bohm phases are invariant only under shifts of φ and $\widetilde{\varphi}$ by integer multiples of $\frac{2\pi}{k}$. This corresponds to an explicit breaking of the original and dual lattice $U(1)$ gauge symmetries down to $\mathbb{Z}(k)$.

5.3 Fate of the Original and Dual $U(1)$ Gauge Symmetries

As we have just seen, when we fix the values of the self-adjoint extension parameters θ and $\widetilde{\theta}$, the original and dual continuous $U(1)$ gauge symmetries of the classical compact doubled Chern-Simons-Maxwell theory are explicitly broken down to discrete $\mathbb{Z}(k)$ gauge symmetries. Since gauge symmetries just reflect redundancies in our theoretical description and are not actual physical symmetries, one may wonder what happened at the quantum level to the redundancy associated with the continuous $U(1)$ gauge symmetries of the classical theory. In order to answer this question, let us perform original and dual continuous $U(1)$ gauge transformations on the link variables

$$\begin{aligned}
U'_{x,i} &= \exp(i\varphi'_{x,i}) = \exp(i\chi_{x-\frac{a}{2}\hat{i}})U_{x,i}\exp(-i\chi_{x+\frac{a}{2}\hat{i}}) \\
&= \exp(i(\varphi_{x,i} - \chi_{x+\frac{a}{2}\hat{i}} + \chi_{x-\frac{a}{2}\hat{i}})), \\
\widetilde{U}'_{x,i} &= \exp(i\widetilde{\varphi}'_{x,i}) = \exp(i\widetilde{\chi}_{x-\frac{a}{2}\hat{i}})\widetilde{U}_{x,i}\exp(-i\widetilde{\chi}_{x+\frac{a}{2}\hat{i}}) \\
&= \exp(i(\widetilde{\varphi}_{x,i} - \widetilde{\chi}_{x+\frac{a}{2}\hat{i}} + \widetilde{\chi}_{x-\frac{a}{2}\hat{i}})).
\end{aligned} \tag{5.16}$$

While the magnetic part of the Hamiltonian is by construction manifestly gauge invariant, the electric part is at least not obviously gauge invariant, because E and $\widetilde{E}$ depend on φ and $\widetilde{\varphi}$. Still, in the non-compact theory, E and $\widetilde{E}$ turn out to be gauge invariant, i.e. they commute with local gauge transformations G and $\widetilde{G}$. Let us return to the compact theory and consider a general finite gauge transformation that is represented in Hilbert space by a unitary transformation

$$V = \prod_{x\in\Lambda} \exp(i\chi_x G_x) \prod_{x\in\widetilde{\Lambda}} \exp(i\widetilde{\chi}_x \widetilde{G}_x). \tag{5.17}$$

It is straightforward to convince oneself that indeed

$$VU_{x,i}V^\dagger = U'_{x,i}, \quad V\widetilde{U}_{x,i}V^\dagger = \widetilde{U}'_{x,i}, \tag{5.18}$$

with $U'_{x,i}$ and $\widetilde{U}'_{x,i}$ given by eq.(5.16). Furthermore, at least at a formal level,

$$VE_{x,i}V^\dagger = E_{x,i}, \quad V\widetilde{E}_{x,i}V^\dagger = \widetilde{E}_{x,i}, \quad VHV^\dagger = H, \tag{5.19}$$

which seems to suggest that the compact theory is still $U(1)$ gauge invariant. However, this is not the case, because V does not leave the domain of the Hamiltonian invariant. In other words, the gauge transformed wave functional $\Psi'[\varphi,\widetilde{\varphi}] = V\Psi[\varphi,\widetilde{\varphi}]$ no longer satisfies the self-adjoint extension condition eq.(5.11). Unlike the wave function $\Psi(\varphi,\widetilde{\varphi})$ (which only depends on the link variables φ and $\widetilde{\varphi}$ of a single cross), the wave functional $\Psi[\varphi,\widetilde{\varphi}]$ depends on the entire lattice gauge fields $[\varphi,\widetilde{\varphi}]$. Under a general gauge transformation, one obtains

$$\Psi'[\varphi,\widetilde{\varphi}] = V\Psi[\varphi,\widetilde{\varphi}] = \prod_{x\in\Lambda} \exp\left(i\frac{ka^2}{4\pi}\chi_x \widetilde{B}_x\right) \prod_{x\in\widetilde{\Lambda}} \exp\left(i\frac{ka^2}{4\pi}\widetilde{\chi}_x B_x\right) \Psi[\varphi',\widetilde{\varphi}'], \tag{5.20}$$

where φ' and $\widetilde{\varphi}'$ are the gauge transformed fields of eq.(5.16), and the plaquette magnetic fields are given by

$$\begin{aligned}
\exp(ia^2B_x) &= \exp(i(\varphi_{x-\frac{a}{2}\hat{2},1} + \varphi_{x+\frac{a}{2}\hat{1},2} - \varphi_{x+\frac{a}{2}\hat{2},1} - \varphi_{x-\frac{a}{2}\hat{1},2})), \quad x \in \widetilde{\Lambda}, \\
\exp(ia^2\widetilde{B}_x) &= \exp(i(\widetilde{\varphi}_{x-\frac{a}{2}\hat{2},1} + \widetilde{\varphi}_{x+\frac{a}{2}\hat{1},2} - \widetilde{\varphi}_{x+\frac{a}{2}\hat{2},1} - \widetilde{\varphi}_{x-\frac{a}{2}\hat{1},2})), \quad x \in \Lambda.
\end{aligned} \tag{5.21}$$

When one applies the unitary transformation to the wave function of a single cross, one finds that the transformed wave function obeys the boundary condition

$$\begin{aligned}
\Psi'(\varphi+2\pi,\widetilde{\varphi}) &= \exp\left(-i\frac{k}{2}\widetilde{\varphi} + i\widetilde{\theta}'\right)\Psi'(\varphi,\widetilde{\varphi}), \\
\Psi'(\varphi,\widetilde{\varphi}+2\pi) &= \exp\left(i\frac{k}{2}\varphi + i\theta'\right)\Psi'(\varphi,\widetilde{\varphi}),
\end{aligned} \tag{5.22}$$

where the self-adjoint extension parameters θ and $\widetilde{\theta}$, which can be associated with the link and its dual link, respectively, have been transformed to

$$\begin{aligned}
\exp(i\theta'_{x,i}) &= \exp(i(\theta_{x,i} + k\chi_{x+\frac{a}{2}\hat{i}} - k\chi_{x-\frac{a}{2}\hat{i}})), \\
\exp(i\widetilde{\theta}'_{x,i}) &= \exp(i(\widetilde{\theta}_{x,i} + k\widetilde{\chi}_{x+\frac{a}{2}\hat{i}} - k\widetilde{\chi}_{x-\frac{a}{2}\hat{i}})).
\end{aligned} \tag{5.23}$$

This means that the self-adjoint extension parameters $\exp(i\theta_{x,i})$ and $\exp(i\widetilde{\theta}_{x,i})$ themselves also transform as $U(1)$ gauge fields. However, unlike $U_{x,i}$ and $\widetilde{U}_{x,i}$, they transport k units of charge. As a consequence, $\exp(i\theta_{x,i})$ and $\exp(i\widetilde{\theta}_{x,i})$ are invariant against $\mathbb{Z}(k)$ gauge transformations. Since a $U(1)$ gauge transformation changes the self-adjoint extension parameters θ and $\widetilde{\theta}$, and thus the domain of the Hamiltonian, it can no longer be considered as a symmetry of the theory. However, the redundancy associated with the $U(1)$ gauge symmetries of the classical theory is still present at the quantum level. Continuous $U(1)$ gauge transformations simply lead from one domain of H (characterized by θ and $\widetilde{\theta}$) to a unitarily equivalent domain characterized by θ' and $\widetilde{\theta}'$. The gauge invariant physical content of the theory is thus not determined by the link parameters $\theta_{x,i}$ and $\widetilde{\theta}_{x,i}$ themselves, but by their plaquette field strengths

$$\exp(i\eta_x) = \exp(i(\theta_{x-\frac{a}{2}\hat{2},1} + \theta_{x+\frac{a}{2}\hat{1},2} - \theta_{x+\frac{a}{2}\hat{2},1} - \theta_{x-\frac{a}{2}\hat{1},2})), \quad x \in \widetilde{\Lambda},$$
$$\exp(i\widetilde{\eta}_x) = \exp(i(\widetilde{\theta}_{x-\frac{a}{2}\hat{2},1} + \widetilde{\theta}_{x+\frac{a}{2}\hat{1},2} - \widetilde{\theta}_{x+\frac{a}{2}\hat{2},1} - \widetilde{\theta}_{x-\frac{a}{2}\hat{1},2})), \quad x \in \Lambda. \quad (5.24)$$

A natural choice is $\eta_x = \widetilde{\eta}_x = 0$ for all x. Another choice is $\eta_x = \widetilde{\eta}_x = \pi$. Both of these choices leave the charge conjugation symmetry C intact, while other choices break C explicitly. The parity symmetry P is already explicitly broken by the Chern-Simons term. It should be pointed out that, unlike $U_{x,i}$ and $\widetilde{U}_{x,i}$, $\exp(i\theta_{x,i})$ and $\exp(i\widetilde{\theta}_{x,i})$ are not dynamical gauge degrees of freedom, but parameters that define a super-selection sector. In this sense, they are analogous to the θ vacuum angle in 4-dimensional non-Abelian gauge theories. In particular, different values of θ define different vacuum sectors. For $\theta \neq 0, \pi$, in non-Abelian gauge theories the vacuum angle explicitly breaks the CP symmetry.

What is the significance of the reduction of the manifest gauge symmetry from $U(1)$ to $\mathbb{Z}(k)$ at the quantum level? In particular, should one interpret this explicit quantum mechanical symmetry breaking as an anomaly? It is important to note that the quantum theory is sensitive to external parameters (the gauge fields θ and $\widetilde{\theta}$ representing Aharonov-Bohm phases) which do not affect the dynamics at the classical level. Remarkably, in this case the external parameters are $\mathbb{Z}(k)$ gauge invariant but change under $U(1)$ gauge transformations.

5.4 Local Magnetic Translation Group on a Single Cross

As we have stressed before, the actual manifest local symmetry of the Hamiltonian in a given domain is reduced to $\mathbb{Z}(k)$. To investigate this symmetry, first let us again consider the purely electric Hamiltonian $H_+ = \frac{e^2a^2}{2}(E^2 + \widetilde{E}^2)$, which commutes with the unitary operators

$$T = \exp\left(\frac{2\pi i}{k}\left(-i\partial_\varphi + \frac{k}{4\pi}\widetilde{\varphi} - \frac{\widetilde{\theta}}{2\pi}\right)\right) = \exp\left(\frac{2\pi}{k}\partial_\varphi + \frac{i}{2}\widetilde{\varphi} - \frac{i}{k}\widetilde{\theta}\right),$$
$$\widetilde{T} = \exp\left(\frac{2\pi i}{k}\left(-i\partial_{\widetilde{\varphi}} - \frac{k}{4\pi}\varphi - \frac{\theta}{2\pi}\right)\right) = \exp\left(\frac{2\pi}{k}\partial_{\widetilde{\varphi}} - \frac{i}{2}\varphi - \frac{i}{k}\theta\right), \quad (5.25)$$

that induce translations by $\frac{2\pi}{k}$ of the wave function (up to an abstract gauge transformation), i.e.

$$
\begin{aligned}
T\Psi(\varphi,\widetilde{\varphi}) &= \exp\left(i\frac{\widetilde{\varphi}}{2}-\frac{i}{k}\widetilde{\theta}\right)\Psi(\varphi+\frac{2\pi}{k},\widetilde{\varphi}),\\
\widetilde{T}\Psi(\varphi,\widetilde{\varphi}) &= \exp\left(-i\frac{\varphi}{2}-\frac{i}{k}\theta\right)\Psi(\varphi,\widetilde{\varphi}+\frac{2\pi}{k}).
\end{aligned}
\tag{5.26}
$$

It is straightforward to rewrite the boundary condition eq.(5.11) as

$$
T^k\Psi(\varphi,\widetilde{\varphi}) = \Psi(\varphi,\widetilde{\varphi}), \quad \widetilde{T}^k\Psi(\varphi,\widetilde{\varphi}) = \Psi(\varphi,\widetilde{\varphi}). \tag{5.27}
$$

This immediately implies that T and $\widetilde{T}$ respect the domain structure of the Hamiltonian, i.e., after acting with these operators, the wave function still obeys the boundary condition eq.(5.11) that defines the domain of H_+. The operators T and $\widetilde{T}$ obey

$$
\widetilde{T}T = \exp\left(\frac{2\pi i}{k}\right)T\widetilde{T}. \tag{5.28}
$$

These two operators generate a discrete group $\mathcal{G}$, known as the magnetic translation group [56, 57], which consists of the elements

$$
g(n,\widetilde{n},m) = \exp\left(\frac{2\pi i m}{k}\right)\widetilde{T}^{\widetilde{n}}T^n, \quad n,\widetilde{n},m\in\{0,1,\ldots,k-1\}. \tag{5.29}
$$

The group multiplication rule is given by

$$
g(n,\widetilde{n},m)g(n',\widetilde{n}',m') = g(n+n',\widetilde{n}+\widetilde{n}',m+m'-n\widetilde{n}'). \tag{5.30}
$$

Here all summations are understood modulo k. The unit element is given by

$$
\mathbb{1} = g(0,0,0), \tag{5.31}
$$

and the elements

$$
z_m = g(0,0,m) = \exp\left(\frac{2\pi i m}{k}\right), \tag{5.32}
$$

form a cyclic Abelian subgroup $\mathbb{Z}(k)\subset\mathcal{G}$. The inverse of the group element $g(n,\widetilde{n},m)$ is given by

$$
g(n,\widetilde{n},m)^{-1} = g(-n,-\widetilde{n},-m-n\widetilde{n}). \tag{5.33}
$$

This follows because

$$
g(n,\widetilde{n},m)g(-n,-\widetilde{n},-m-n\widetilde{n}) = g(0,0,-n\widetilde{n}+n\widetilde{n}) = g(0,0,0) = \mathbb{1}. \tag{5.34}
$$

Let us now consider the conjugacy class of a group element $g(n,\widetilde{n},m)$ that consists of the elements

$$
\begin{aligned}
&g(n',\widetilde{n}',m')g(n,\widetilde{n},m)g(n',\widetilde{n}',m')^{-1} =\\
&g(n'+n,\widetilde{n}'+\widetilde{n},m'+m-n'\widetilde{n})g(-n',-\widetilde{n}',-m'-n'\widetilde{n}') =\\
&g(n,\widetilde{n},m-n'(\widetilde{n}+\widetilde{n}')+(n'+n)\widetilde{n}') = g(n,\widetilde{n},m+n\widetilde{n}'-n'\widetilde{n}).
\end{aligned}
\tag{5.35}
$$

The elements $g(0, 0, m) = z_m \in \mathbb{Z}(k)$ are conjugate only to themselves and thus form k single-element conjugacy classes. Multiplication by a phase z_m amounts to a $\mathbb{Z}(k)$ gauge transformation. The conjugacy classes hence correspond to gauge equivalence classes. The elements $g(0, 0, m) = z_m$ commute with all other elements and thus form the center $\mathbb{Z}(k)$ of the group $\mathcal{G}$. Since the individual elements of the center form separate conjugacy classes, the center forms a normal subgroup which can be factored out. This corresponds to identifying gauge equivalence classes. It should be pointed out that $\mathcal{G}$ is not the direct product $\mathbb{Z}(k) \times \mathbb{Z}(k) \times \mathbb{Z}(k)$. Since the quotient space $\mathcal{G}/\mathbb{Z}(k)$ is not a subgroup of $\mathcal{G}$, $\mathcal{G}$ is not even a semi-direct product of $\mathbb{Z}(k) \times \mathbb{Z}(k)$ and $\mathbb{Z}(k)$, but just a particular central extension of $\mathbb{Z}(k) \times \mathbb{Z}(k)$ by the center subgroup $\mathbb{Z}(k)$.

5.5 Spectrum of the Hamiltonian on a Single Cross

Before we consider the full Hamiltonian of the entire lattice theory, let us again consider the purely electric Hamiltonian H_+ of a single cross. It commutes with both T and $\widetilde{T}$, which, however, don't commute with each other. Let us construct simultaneous eigenstates of H_+ and T. Since $T^k = \mathbb{1}$, the eigenvalues of T are given by $\exp(2\pi i l/k)$ with $l \in \{0, 1, \ldots, k-1\}$, while the eigenvalues of H_+ are given by $E_n = M(n + \frac{1}{2})$ (where $n \in \{0, 1, 2, \ldots\}$ denotes the Landau level [58, 59] in the "mechanical" analog, and M is the "photon" mass). Hence, we can construct simultaneous eigenstates $|nl\rangle$ such that

$$H_+|nl\rangle = M\left(n + \frac{1}{2}\right)|nl\rangle, \; T|nl\rangle = \exp\left(\frac{2\pi i l}{k}\right)|nl\rangle. \tag{5.36}$$

As a consequence of eq.(5.28) one obtains

$$T\widetilde{T}|nl\rangle = \exp\left(-\frac{2\pi i}{k}\right)\widetilde{T}T|nl\rangle = \exp\left(\frac{2\pi i(l-1)}{k}\right)\widetilde{T}|nl\rangle, \tag{5.37}$$

which implies

$$\widetilde{T}|nl\rangle = |n(l-1)\rangle. \tag{5.38}$$

Since $[\widetilde{T}, H] = 0$, the k states $|nl\rangle$ with $l \in 0, 1, \ldots, k-1$ are degenerate and form an irreducible representation of the magnetic translation group. If we restrict ourselves to the lowest Landau level, $n = 0$, by sending $e \to \infty$ such that $M \to \infty$, the local Hilbert space of a cross is k-dimensional. As we will see later, this is the limit in which the doubled Chern-Simons-Maxwell theory reduces to the $\mathbb{Z}(k)$ variant of the toric code.

In order to prepare for the derivation of the toric code in a later subsection, it is useful to explicitly construct the eigenfunctions $|nl\rangle$, which are given by

$$\begin{aligned} \Psi_{nl}^{\theta,\widetilde{\theta}}(\varphi,\widetilde{\varphi}) &= \langle\varphi\widetilde{\varphi}|nl\rangle = \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\psi_n\left(\widetilde{\varphi}-2\pi\left[m+\frac{l}{k}+\frac{\widetilde{\theta}}{2\pi k}\right]\right) \\ &\times \ \exp\left(i(k\varphi+\theta)\left[m+\frac{l}{k}+\frac{\widetilde{\theta}}{2\pi k}\right]-i\frac{k\varphi\widetilde{\varphi}}{4\pi}\right). \end{aligned} \tag{5.39}$$

Here

$$\psi_n(\widetilde{\varphi}) = \sqrt[4]{\frac{k}{2\pi}}\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\exp\left(-\frac{k}{2\pi}\frac{\widetilde{\varphi}^2}{2}\right)H_n\left(\sqrt{\frac{k}{2\pi}}\widetilde{\varphi}\right), \tag{5.40}$$

(with H_n denoting the Hermite polynomials) are the eigenfunctions of a 1-dimensional harmonic oscillator with characteristic "frequency" given by the "photon" mass $M = \frac{ke^2}{2\pi}$. The corresponding energy eigenvalue is thus given by $E_n = M(n+\frac{1}{2})$. It is straightforward to convince oneself that these wave functions indeed satisfy the correct boundary conditions eq.(5.11) and are eigenstates of H_+. The boundary conditions on the torus parametrized by the self-adjoint extension parameters θ and $\widetilde{\theta}$ are given by

$$\begin{aligned} \Psi_{nl}^{\theta+2\pi,\widetilde{\theta}}(\varphi,\widetilde{\varphi}) &= \exp\left(i\frac{2\pi l+\widetilde{\theta}}{k}\right)\Psi_{nl}^{\theta,\widetilde{\theta}}(\varphi,\widetilde{\varphi}) = \Omega_{ll'}\Psi_{nl'}^{\theta,\widetilde{\theta}}(\varphi,\widetilde{\varphi}), \\ \Psi_{nl}^{\theta,\widetilde{\theta}+2\pi}(\varphi,\widetilde{\varphi}) &= \Psi_{n,l+1}^{\theta,\widetilde{\theta}}(\varphi,\widetilde{\varphi}) = \widetilde{\Omega}_{ll'}\Psi_{nl'}^{\theta,\widetilde{\theta}}(\varphi,\widetilde{\varphi}). \end{aligned} \tag{5.41}$$

The boundary conditions of eq.(5.41) take the form of 't Hooft's $U(k)$ twisted boundary conditions [60, 61]. The transition functions $\Omega, \widetilde{\Omega} \in U(k)$ play the role of $U(k)$ gauge transformations. Their matrix elements are given by

$$\Omega_{ll'} = \exp\left(i\frac{2\pi l+\widetilde{\theta}}{k}\right)\delta_{ll'}, \quad \widetilde{\Omega}_{ll'} = \delta_{l+1,l'}, \tag{5.42}$$

where $\delta_{l+1,l'}$ is understood modulo k. The transition functions satisfy the cocycle consistency condition

$$\Omega\widetilde{\Omega} = \exp\left(\frac{2\pi i}{k}\right)\widetilde{\Omega}\Omega, \tag{5.43}$$

which is characterized by the $U(k)$ gauge invariant twist factor $\exp(2\pi i/k) \in \mathbb{Z}(k)$. The twist factor is an element of the center of $U(k)$. The operators Ω and $\widetilde{\Omega}$ obey the same commutation relation as $\widetilde{T}$ and T (cf. eq.(5.28)) and thus also generate a representation of the magnetic translation group. As $\widetilde{\theta}$ is increased by 2π, the $\mathbb{Z}(k)$ electric flux quantum number l increases by 1. An adiabatic change of the self-adjoint extension parameters thus allows us to continuously interpolate between the different discrete $\mathbb{Z}(k)$ electric flux sectors.

5.6 $U(k)$ Berry Gauge Fields

It is natural to ask how the wave functions on a single cross respond to an adiabatic change of θ and $\widetilde{\theta}$. Since k states $|nl\rangle$ (with $l \in \{0, 1, \ldots, k-1\}$) are degenerate, this gives rise to an abstract $U(k)$ non-Abelian Berry gauge field on the torus $U(1)^2$ parametrized by θ and $\widetilde{\theta}$,

$$\begin{aligned} g_{ll'}(\theta, \widetilde{\theta}) &= \langle nl|W^\dagger \partial_\theta W|nl'\rangle = 0, \quad W = \exp\left(-\frac{i}{2\pi}(\theta\widetilde{\varphi} + \widetilde{\theta}\varphi)\right), \\ \widetilde{g}_{ll'}(\theta, \widetilde{\theta}) &= \langle nl|W^\dagger \partial_{\widetilde{\theta}} W|nl'\rangle = i\frac{\theta}{2\pi k}\delta_{ll'}. \end{aligned} \tag{5.44}$$

In order to account for the fact that different values of θ and $\widetilde{\theta}$ correspond to different domains of the Hamiltonian, it is important to include the unitary transformation W in the definition of the Berry connection. The $g_{ll'}$ and $\widetilde{g}_{ll'}$ are elements of two $k \times k$ anti-Hermitean matrix-valued fields g and $\widetilde{g}$ that play the role of abstract $U(k)$ Berry vector potentials. As we see, g turns out to be Abelian, and $\widetilde{g}$ even vanishes, at least in the gauge that we have picked. The corresponding Berry field strength, i.e. the abstract "magnetic" field, is given by

$$h(\theta, \widetilde{\theta}) = \partial_\theta \widetilde{g}(\theta, \widetilde{\theta}) - \partial_{\widetilde{\theta}} g(\theta, \widetilde{\theta}) + [g(\theta, \widetilde{\theta}), \widetilde{g}(\theta, \widetilde{\theta})], \quad h_{ll'}(\theta, \widetilde{\theta}) = \frac{i}{2\pi k}\delta_{ll'}. \tag{5.45}$$

It is again Abelian and constant over the 2-dimensional torus parametrized by θ and $\widetilde{\theta}$. Since the field strength is proportional to the unit-matrix, it is even invariant under general non-Abelian $U(k)$ gauge transformations (resulting from a basis change in the subspace of degenerate states $|nl\rangle$). This means that it actually reduces to an Abelian $U(1)$ Berry field strength. Still, since the field strength is non-zero, it can give rise to non-trivial Berry phases. However, since it is Abelian, its use for quantum information processing will be limited. The total "magnetic" flux that threads the torus is given by $\frac{2\pi}{k}$. This is consistent with the twist factor $\exp(2\pi i/k) \in \mathbb{Z}(k)$ of eq.(5.43) that characterizes 't Hooft's twisted boundary condition [60, 61].

Besides the Berry field strength, there are gauge invariant Berry Polyakov loops that result from the holonomies of the torus parametrized by the periodic self-adjoint extension parameters $(\theta, \widetilde{\theta})$. By integrating the Berry connection along a straight line wrapping around the torus, we obtain

$$\int_0^{2\pi} d\theta \; g_{ll}(\theta, \widetilde{\theta}) = 0, \quad \int_0^{2\pi} d\widetilde{\theta} \; \widetilde{g}_{ll}(\theta, \widetilde{\theta}) = i\frac{\theta}{k}. \tag{5.46}$$

The Polyakov loop matrices Φ, $\widetilde{\Phi} \in U(k)$, which describe parallel transport around the torus, also receive contributions from the transition functions Ω and $\widetilde{\Omega}$ such that

$$\Phi_{ll'}(\widetilde{\theta}) = \Omega_{ll'} = \exp\left(i\frac{2\pi l + \widetilde{\theta}}{k}\right)\delta_{ll'}, \quad \widetilde{\Phi}_{ll'}(\theta) = \exp\left(i\frac{\theta}{k}\right)\widetilde{\Omega}_{ll'} = \exp\left(i\frac{\theta}{k}\right)\delta_{l+1,l'}. \tag{5.47}$$

As we will see later, this will allow us to derive the mutual statistics angle $\frac{2\pi}{k}$ also in the compact Chern-Simons theory.

5.7 $\mathbb{Z}(k)$ Gauge Symmetry and Gauss Law

The operators T and $\widetilde{T}$ are associated with the links of a given cross. Until now we have considered a cross centered at $x \in X$. Now we also add the corresponding objects for $x \in \widetilde{X}$ such that

$$\begin{aligned} T_{x,i} &= \exp\left(\frac{2\pi}{k}\partial_{\varphi_{x,i}} + \frac{i}{2}\epsilon_{ij}\widetilde{\varphi}_{x,j} - \frac{i}{k}\epsilon_{ij}\widetilde{\theta}_{x,j}\right), \\ \widetilde{T}_{x,i} &= \exp\left(\frac{2\pi}{k}\partial_{\widetilde{\varphi}_{x,i}} + \frac{i}{2}\epsilon_{ij}\varphi_{x,j} + \frac{i}{k}\epsilon_{ij}\theta_{x,j}\right). \end{aligned} \tag{5.48}$$

A $\mathbb{Z}(k)$ Gauss law combines T or $\widetilde{T}$ operators from links that touch a common lattice point to form unitary operators that represent local $\mathbb{Z}(k)$ gauge transformations

$$V_x = \prod_i T_{x+\frac{a}{2}\hat{i},i} T^\dagger_{x-\frac{a}{2}\hat{i},i},\ x \in \Lambda, \quad \widetilde{V}_x = \prod_i \widetilde{T}_{x+\frac{a}{2}\hat{i},i} \widetilde{T}^\dagger_{x-\frac{a}{2}\hat{i},i},\ x \in \widetilde{\Lambda}. \tag{5.49}$$

By construction, V_x and $\widetilde{V}_x$ represent the manifest $\mathbb{Z}(k)$ gauge symmetries of the full Hamiltonian (including both the electric cross and the magnetic plaquette contributions), which respect the domain structure defined by the values of the self-adjoint extension parameters θ and $\widetilde{\theta}$. Although T and $\widetilde{T}$ on the same cross do not commute, original and dual gauge transformations on neighboring sites commute (cf. Fig.3)

$$\begin{aligned} & V_x \widetilde{V}_{x+\frac{a}{2}\hat{1}+\frac{a}{2}\hat{2}} = \\ & T_{x+\frac{a}{2}\hat{1},1} T_{x+\frac{a}{2}\hat{2},2} T^\dagger_{x-\frac{a}{2}\hat{1},1} T^\dagger_{x-\frac{a}{2}\hat{2},2} \widetilde{T}_{x+a\hat{1}+\frac{a}{2}\hat{2},1} \widetilde{T}_{x+\frac{a}{2}\hat{1}+a\hat{2},2} \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{2},1} \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{1},2} = \\ & \widetilde{T}_{x+a\hat{1}+\frac{a}{2}\hat{2},1} \widetilde{T}_{x+\frac{a}{2}\hat{1}+a\hat{2},2} \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{2},1} \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{1},2} T_{x+\frac{a}{2}\hat{1},1} T_{x+\frac{a}{2}\hat{2},2} T^\dagger_{x-\frac{a}{2}\hat{1},1} T^\dagger_{x-\frac{a}{2}\hat{2},2} = \widetilde{V}_{x+\frac{a}{2}\hat{1}+\frac{a}{2}\hat{2}} V_x. \end{aligned} \tag{5.50}$$

Here we have used

$$\begin{aligned} T_{x+\frac{a}{2}\hat{1},1} \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{1},2} &= \exp\left(\frac{2\pi i}{k}\right) \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{1},2} T_{x+\frac{a}{2}\hat{1},1}, \\ T_{x+\frac{a}{2}\hat{2},2} \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{2},1} &= \exp\left(-\frac{2\pi i}{k}\right) \widetilde{T}^\dagger_{x+\frac{a}{2}\hat{2},1} T_{x+\frac{a}{2}\hat{2},2}. \end{aligned} \tag{5.51}$$

For given θ and $\widetilde{\theta}$, physical states without external charges from the corresponding domain of H must obey the $\mathbb{Z}(k)$ Gauss law

$$V_x|\Psi\rangle = |\Psi\rangle,\ x \in \Lambda, \quad \widetilde{V}_x|\Psi\rangle = |\Psi\rangle,\ x \in \widetilde{\Lambda}. \tag{5.52}$$

States that have external charges $Q_x, \widetilde{Q}_x \in 0, 1, \ldots, k-1$ on the original and dual lattice obey

$$V_x|Q, \widetilde{Q}\rangle = \exp\left(\frac{2\pi i}{k} Q_x\right)|Q, \widetilde{Q}\rangle, \quad x \in \Lambda,$$
$$\widetilde{V}_x|Q, \widetilde{Q}\rangle = \exp\left(\frac{2\pi i}{k} \widetilde{Q}_x\right)|Q, \widetilde{Q}\rangle, \quad x \in \widetilde{\Lambda}. \tag{5.53}$$

As we have seen before, θ and $\widetilde{\theta}$ represent external non-dynamical $U(1)$ lattice gauge fields. When we perform $U(1)$ gauge transformations, we may change the values of θ and $\widetilde{\theta}$, which leads us into a unitarily equivalent domain of H, thus leaving the physics invariant. The physics does, however, depend on the gauge invariant plaquette field strengths $\exp(i\eta_x)$ and $\exp(i\widetilde{\eta}_x)$ of eq.(5.24).

5.8 Pure Chern-Simons Limit

Let us now take the limit of large e^2, which implies that the "photon" of mass $M = \frac{ke^2}{2\pi}$ is removed from the spectrum and the system reduces to a pure Chern-Simons theory. In particular, the low-energy physics is restricted to the lowest Landau level $n = 0$. Then there are only k states left on each individual cross. As before, we choose an electric flux basis of the local Hilbert space, characterized by the fluxes $l \in \{0, 1, \ldots, k-1\}$ on the links of the original lattice. The electric field energy is then given by $\frac{M}{2}$ for each cross, and the dynamics is entirely controlled by the magnetic field energy as well as by the Gauss law. Since they are suppressed by $\frac{1}{e^2}$, the magnetic contributions to the energy can then be treated in first order degenerate perturbation theory, by evaluating their matrix elements between the degenerate electric flux eigenstates $|0l\rangle$. Putting $n = 0$ in eq.(5.39), it is straightforward to obtain

$$\begin{aligned}
\langle 0l|\exp(\pm i\varphi)|0l'\rangle &= C\exp\left(\pm\frac{i\theta}{k}\right)\delta_{l,l'\pm 1} = C\exp\left(\pm\frac{i\theta}{k}\right)\langle 0l|\widetilde{T}^{\mp 1}|0l'\rangle, \\
\langle 0l|\exp(\pm i\widetilde{\varphi})|0l'\rangle &= C\exp\left(\pm\frac{i\widetilde{\theta}}{k}\right)\exp\left(\pm\frac{2\pi i l}{k}\right)\delta_{ll'} \\
&= C\exp\left(\pm\frac{i\widetilde{\theta}}{k}\right)\langle 0l|T^{\pm 1}|0l'\rangle, \quad C = \exp\left(-\frac{\pi}{2k}\right).
\end{aligned} \tag{5.54}$$

Using $\widetilde{T}|0l\rangle = |0(l-1)\rangle$, this implies that (up to a trivial additive constant) the effective low-energy Hamiltonian (in the $n = 0$ sector) is given by

$$\begin{aligned}
H_{\text{eff}} &= -\frac{C^4}{2e^2a^2}\sum_{x\in\widetilde{\Lambda}}\left[\exp\left(\frac{i\eta_x}{k}\right)\widetilde{V}_x + \exp\left(-\frac{i\eta_x}{k}\right)\widetilde{V}_x^\dagger\right] \\
&- \frac{C^4}{2e^2a^2}\sum_{x\in\Lambda}\left[\exp\left(\frac{i\widetilde{\eta}_x}{k}\right)V_x^\dagger + \exp\left(-\frac{i\widetilde{\eta}_x}{k}\right)V_x\right],
\end{aligned} \tag{5.55}$$

where η and $\widetilde{\eta}$ are the plaquette field strength variables associated with the non-dynamical gauge fields θ and $\widetilde{\theta}$ of eq.(5.24). The first contribution results from a $\cos(a^2 B_x)$ term on the original plaquettes and the second contribution originates from a dual plaquette term $\cos(a^2 \widetilde{B}_x)$. When the Hamiltonian acts on eigenstates $|Q, \widetilde{Q}\rangle$ with charges $Q_x, \widetilde{Q}_x$ on the original and on the dual lattice (cf. eq.(5.53)) one obtains

$$H_{\text{eff}}|Q,\widetilde{Q}\rangle = -\frac{C^4}{e^2 a^2}\left[\sum_{x\in\widetilde{\Lambda}} \cos\left(\frac{2\pi\widetilde{Q}_x + \eta_x}{k}\right) + \sum_{x\in\Lambda} \cos\left(\frac{2\pi Q_x - \widetilde{\eta}_x}{k}\right)\right]|Q,\widetilde{Q}\rangle. \tag{5.56}$$

Hence, the effective Hamiltonian punishes violations of the $\mathbb{Z}(k)$ Gauss law on the original and on the dual lattice. In particular, for $\eta_x = \widetilde{\eta}_x = 0$, the ground state is reached when $V_x|\Psi\rangle = |\Psi\rangle$ for $x \in \Lambda$ and $\widetilde{V}_x|\Psi\rangle = |\Psi\rangle$ for $x \in \widetilde{\Lambda}$, i.e. when $Q_x = \widetilde{Q}_x = 0$. For other values of η_x and $\widetilde{\eta}_x$ other constellations of charges are energetically favored.

Let us construct the states $|Q, \widetilde{Q}\rangle$ more explicitly. For this purpose, we construct dual charge projection operators at each site $x \in \widetilde{\Lambda}$

$$P_{\widetilde{Q}_x} = \frac{1}{k}\sum_{m=0}^{k-1} \exp\left(-\frac{2\pi i m}{k}\widetilde{Q}_x\right)\widetilde{V}_x^m, \tag{5.57}$$

which obey

$$P_{\widetilde{Q}_x} P_{\widetilde{Q}'_x} = \delta_{\widetilde{Q}_x,\widetilde{Q}'_x} P_{\widetilde{Q}_x}, \quad \widetilde{V}_x P_{\widetilde{Q}_x} = \exp\left(\frac{2\pi i}{k}\widetilde{Q}_x\right) P_{\widetilde{Q}_x}. \tag{5.58}$$

We now consider a state $|[l]\rangle$ corresponding to a specific configuration $[l]$ of electric fluxes that obeys the Gauss law in the presence of charges $Q_x \in \mathbb{Z}(k)$ on the original lattice, $x \in \Lambda$, i.e.

$$\sum_i (l_{x+\frac{a}{2}\hat{i},i} - l_{x-\frac{a}{2}\hat{i},i}) \text{mod} k = Q_x. \tag{5.59}$$

By construction, we then have

$$V_x|[l]\rangle = \exp\left(\frac{2\pi i}{k} Q_x\right)|[l]\rangle, \quad x \in \Lambda. \tag{5.60}$$

Next we apply the charge projection operators at all dual sites to obtain

$$|Q,\widetilde{Q}\rangle = \mathcal{N}\prod_{x\in\widetilde{\Lambda}} P_{\widetilde{Q}_x}|[l]\rangle, \tag{5.61}$$

where $\mathcal{N}$ is a normalization factor. By construction, this state indeed obeys

$$\widetilde{V}_x|Q,\widetilde{Q}\rangle = \exp\left(\frac{2\pi i}{k}\widetilde{Q}_x\right)|Q,\widetilde{Q}\rangle, \quad x \in \widetilde{\Lambda}. \tag{5.62}$$

As we have seen in Fig.4, in the non-compact theory the transport of a charge along a curve $\mathcal{C} = \partial S$ on the original lattice, that encircles a total charge $\widetilde{Q}_S = \sum_{x \in S \subset \widetilde{\Lambda}} \widetilde{Q}_x$ on the dual lattice, is associated with a non-trivial topological phase $\exp(2\pi i \widetilde{Q}_S/k)$, thus showing that original and dual charges have mutually anyonic statistics with the statistics angle $\frac{2\pi}{k}$. We will now derive the same result in the compact theory. For this purpose we again consider a Wilson loop

$$W_{\mathcal{C}} = \prod_{(x,i) \in \mathcal{C}} U_{x,i} = \prod_{(x,i) \in \mathcal{C}} \exp(i\varphi_{x,i}). \tag{5.63}$$

Using eq.(5.54) we then obtain

$$W_{\mathcal{C}}|Q, \widetilde{Q}\rangle = C^{L_{\mathcal{C}}} \sum_{x \in S \subset \widetilde{\Lambda}} \widetilde{V}_x \prod_{(x,i) \in \mathcal{C}} \exp(i\frac{\theta_{x,i}}{k})|Q, \widetilde{Q}\rangle = C^{L_{\mathcal{C}}} \exp\left(\frac{2\pi i \widetilde{Q}_S}{k} + i\frac{\eta_S}{k}\right)|Q, \widetilde{Q}\rangle. \tag{5.64}$$

Here $L_{\mathcal{C}}$ is the length of the closed loop $\mathcal{C}$ and $\widetilde{Q}_S = \sum_{x \in S \subset \widetilde{\Lambda}} \widetilde{Q}_x$ is the total dual charge encircled by $\mathcal{C}$. As before, the phase of the Wilson loop contains the $\widetilde{Q}_S$-term which implies that original and dual charges have mutually anyonic statistics with the statistics angle $\frac{2\pi}{k}$. Due to the non-trivial background gauge field θ, there is an additional phase determined by the total background flux $\eta_S = \sum_{x \in S \subset \widetilde{\Lambda}} \eta_x$ encircled by $\mathcal{C}$.

The Wilson loop describes the instantaneous transport of a charge around the loop $\mathcal{C}$. Alternatively, we now want to consider a much slower adiabatic process in which a charge-anti-charge pair is created, transported around the loop $\mathcal{C}$, and finally annihilated. For this purpose, we compute the Berry phase that we accumulate when we gradually change the background field $\widetilde{\theta}$ from 0 to 2π on all dual links intersecting the loop $\mathcal{C}$. According to eq.(5.47) this provides us with a factor $\exp(i\theta_{x,i}/k)$ for all links that belong to $\mathcal{C}$. By Stokes theorem, this again yields the phase $\exp(i\eta_S/k)$. In addition, the factor $\delta_{l+1,l'}$ in eq.(5.47) shifts all electric flux variables along $\mathcal{C}$ by 1, which again gives rise to the phase $\exp(2\pi i \widetilde{Q}_S/k)$ when one returns to the initial state $|Q, \widetilde{Q}\rangle$ at the end of the adiabatic charge transport process. This shows explicitly that the anyon statistics angle $\frac{2\pi}{k}$ is insensitive to the details of how the charge transport process is realized, be it adiabatic or instantaneous.

5.9 Relation to the Toric Code

The toric code is a $\mathbb{Z}(2)$ lattice gauge theory with a degenerate ground state that can be used as a storage device for quantum information that is topologically protected against decoherence. The fundamental degrees of freedom on each link of a square lattice are quantum spins $\frac{1}{2}$. In fact, the toric code is a simple member of a large class of unconventional lattice gauge theories known as quantum link models [62–66].

The Hamiltonian of the $\mathbb{Z}(2)$ toric code is given by

$$H = -J \sum_{x \in \widetilde{\Lambda}} U_x - G \sum_{x \in \Lambda} V_x, \quad V_x = \prod_i \exp\left(i\pi S^3_{x+\frac{a}{2}\hat{i},i}\right) \exp\left(-i\pi S^3_{x-\frac{a}{2}\hat{i},i}\right). \quad (5.65)$$

There is a spin $\frac{1}{2}$, $\vec{S}_{x,i}$, associated with each link connecting the neighboring lattice sites $x - \frac{a}{2}\hat{i}$ and $x + \frac{a}{2}\hat{i}$. The quantum link operator is given by $U_{x,i} = S^1_{x,i}$, such that the plaquette term

$$U_x = S^1_{x-\frac{a}{2}\hat{2},1} S^1_{x+\frac{a}{2}\hat{1},2} S^1_{x+\frac{a}{2}\hat{2},1} S^1_{x-\frac{a}{2}\hat{1},2}, \quad x \in \widetilde{\Lambda}, \quad (5.66)$$

is invariant under discrete $\mathbb{Z}(2)$ gauge transformations $\Omega_x = \pm 1$. The G-term with $G > 0$ punishes violations of the $\mathbb{Z}(2)$ Gauss law $V_x|\Psi\rangle = |\Psi\rangle$. Note that for $\mathbb{Z}(2)$ one has $U_x = U_x^\dagger$ and $V_x = V_x^\dagger$.

One can describe the dynamics of the toric code in a basis of electric flux states on each link, which are characterized by the 3-component $S^3_{x,i} = \pm\frac{1}{2}$ of the spin. The two spin values then correspond to two directions of the electric flux on each link. The quantum link operator $U_{x,i} = S^1_{x,i}$ reverses the direction of the electric flux, and thus the plaquette operator U_x reverses the direction of all electric fluxes encircling the plaquette. The $\mathbb{Z}(2)$ toric code naturally generalizes to a $\mathbb{Z}(k)$ variant with k distinct values of the electric flux on each link. The corresponding quantum link operator then increases the flux by one unit modulo k. The $\mathbb{Z}(k)$ Gauss law ensures that the fluxes entering a common lattice point add up to zero (again modulo k). Interestingly, the $\mathbb{Z}(k)$ variant of the toric code is exactly what emerged from the doubled compact Chern-Simons-Maxwell theory in the pure Chern-Simons limit of large "photon" mass, when we identify $J = G = \frac{C^4}{e^2}$.

5.10 Rotating θ and $\widetilde{\theta}$ into the Hamiltonian

Until now, the external background gauge fields θ and $\widetilde{\theta}$ entered the theory via the boundary conditions eq.(5.11) that define the domain of the Hamiltonian. For completeness we would now like to show that the background fields can be rotated into the Hamiltonian by the unitary transformation

$$W = \prod_{x \in X} \exp\left(-\frac{i}{2\pi}(\theta_{x,1}\widetilde{\varphi}_{x,2} + \widetilde{\theta}_{x,2}\varphi_{x,1})\right) \prod_{x \in \widetilde{X}} \exp\left(\frac{i}{2\pi}(\theta_{x,2}\widetilde{\varphi}_{x,1} + \widetilde{\theta}_{x,1}\varphi_{x,2})\right),$$
$$H' = WHW^\dagger, \quad |\Psi'\rangle = W|\Psi\rangle. \quad (5.67)$$

The original Hamiltonian

$$
\begin{aligned}
H &= \frac{e^2}{2}\sum_{x\in X}\left[\left(-i\partial_{\varphi_{x,1}}-\frac{k}{4\pi}\widetilde{\varphi}_{x,2}\right)^2+\left(-i\partial_{\widetilde{\varphi}_{x,2}}+\frac{k}{4\pi}\varphi_{x,1}\right)^2\right] \\
&+ \frac{e^2}{2}\sum_{x\in \widetilde{X}}\left[\left(-i\partial_{\varphi_{x,2}}+\frac{k}{4\pi}\widetilde{\varphi}_{x,1}\right)^2+\left(-i\partial_{\widetilde{\varphi}_{x,1}}-\frac{k}{4\pi}\varphi_{x,2}\right)^2\right] \\
&- \frac{1}{e^2a^2}\sum_{x\in\Lambda}\cos(a^2\widetilde{B}_x)-\frac{1}{e^2a^2}\sum_{x\in\widetilde{\Lambda}}\cos(a^2 B_x),
\end{aligned}
\tag{5.68}
$$

with the boundary condition (for a cross with $x \in X$)

$$
\Psi(\varphi+2\pi,\widetilde{\varphi})=\exp\left(-i\frac{k}{2}\widetilde{\varphi}+i\widetilde{\theta}\right)\Psi(\varphi,\widetilde{\varphi}),\ \Psi(\varphi,\widetilde{\varphi}+2\pi)=\exp\left(i\frac{k}{2}\varphi+i\theta\right)\Psi(\varphi,\widetilde{\varphi}),
\tag{5.69}
$$

then turns into the new Hamiltonian

$$
\begin{aligned}
H' &= \frac{e^2}{2}\sum_{x\in X}\left[\left(-i\partial_{\varphi_{x,1}}-\frac{k}{4\pi}\widetilde{\varphi}_{x,2}+\frac{\widetilde{\theta}_{x,2}}{2\pi}\right)^2+\left(-i\partial_{\widetilde{\varphi}_{x,2}}+\frac{k}{4\pi}\varphi_{x,1}+\frac{\theta_{x,1}}{2\pi}\right)^2\right] \\
&+ \frac{e^2}{2}\sum_{x\in \widetilde{X}}\left[\left(-i\partial_{\varphi_{x,2}}+\frac{k}{4\pi}\widetilde{\varphi}_{x,1}-\frac{\widetilde{\theta}_{x,1}}{2\pi}\right)^2+\left(-i\partial_{\widetilde{\varphi}_{x,1}}-\frac{k}{4\pi}\varphi_{x,2}-\frac{\theta_{x,2}}{2\pi}\right)^2\right] \\
&- \frac{1}{e^2a^2}\sum_{x\in\Lambda}\cos(a^2\widetilde{B}_x)-\frac{1}{e^2a^2}\sum_{x\in\widetilde{\Lambda}}\cos(a^2 B_x),
\end{aligned}
\tag{5.70}
$$

The new Hamiltonian depends on θ and $\widetilde{\theta}$, while the new wave function now obeys the simplified boundary condition (again for a cross with $x \in X$)

$$
\Psi'(\varphi+2\pi,\widetilde{\varphi})=\exp\left(-i\frac{k}{2}\widetilde{\varphi}\right)\Psi'(\varphi,\widetilde{\varphi}),\quad \Psi'(\varphi,\widetilde{\varphi}+2\pi)=\exp\left(i\frac{k}{2}\varphi\right)\Psi'(\varphi,\widetilde{\varphi}).
\tag{5.71}
$$

which is θ- and $\widetilde{\theta}$-independent. The wave function on a single cross $x \in X$ then takes the form

$$
\begin{aligned}
\Psi'^{\theta,\widetilde{\theta}}_{nl}(\varphi,\widetilde{\varphi}) &= \langle\varphi\widetilde{\varphi}|nl\rangle=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\psi_n\left(\widetilde{\varphi}-2\pi\left[m+\frac{l}{k}+\frac{\widetilde{\theta}}{2\pi k}\right]\right) \\
&\times\ \exp\left(i\varphi\left[km+l-\frac{k\widetilde{\varphi}}{4\pi}\right]+i\left[m+\frac{l}{k}+\frac{\widetilde{\theta}}{2\pi k}-\frac{\widetilde{\varphi}}{2\pi}\right]\theta\right).
\end{aligned}
\tag{5.72}
$$

In this case, in contrast to eq.(5.44), and using $|nl\rangle'=W|nl\rangle$, the definition of the Berry connection no longer requires the insertion of W and is simply given by

$$
g_{ll'}(\theta,\widetilde{\theta})={}'\langle nl|\partial_\theta|nl'\rangle',\quad \widetilde{g}_{ll'}(\theta,\widetilde{\theta})={}'\langle nl|\partial_{\widetilde{\theta}}|nl'\rangle'.
\tag{5.73}
$$

The resulting values of the Berry connection are the same as in eq.(5.44).

6 Conclusions

We have obtained the $\mathbb{Z}(k)$ variant of the toric code in the infinite "photon" mass limit of a doubled $U(1)$ Chern-Simons-Maxwell lattice gauge theory. In contrast to ordinary lattice gauge theories, in this case the field algebra is not link-based. Instead it is based on a cross formed by a link and its corresponding dual link. In ordinary lattice gauge theory, each individual link has a "mechanical" analog: a "particle" moving in the group space, i.e. a "particle" moving on a circle for a compact Abelian $U(1)$ lattice gauge theory. In the cross-based doubled compact Chern-Simons-Maxwell theory, on the other hand, the "mechanical" analog is a charged "particle" moving on a torus $U(1)^2$ which is threaded by an abstract "magnetic" flux, determined by the prefactor k of the Chern-Simons term. The Dirac quantization condition for the abstract "magnetic" flux then leads to the quantization of the level $k \in \mathbb{Z}$.

Remarkably, the cross-based electric contribution to the Hamiltonian has two self-adjoint extension parameters θ and $\widetilde{\theta}$, which themselves form two non-dynamical background $U(1)$ lattice gauge fields. In a fixed background, the manifest gauge symmetry of the compact Chern-Simons-Maxwell theory is then reduced from $U(1)$ to $\mathbb{Z}(k)$. In particular, only a $\mathbb{Z}(k)$ (and not the full $U(1)$) Gauss law must be imposed on physical states in a given domain of the Hamiltonian (which is determined by θ and $\widetilde{\theta}$). The explicit symmetry breaking from $U(1)$ to $\mathbb{Z}(k)$ has a quantum mechanical origin, because only the quantum but not the classical theory is sensitive to the external self-adjoint extension parameters, which manifest themselves as Aharonov-Bohm phases. One might thus describe the quantum mechanical gauge symmetry breaking as an "anomaly". However, it is more like the explicit breaking of CP invariance due to a non-zero θ-vacuum angle in 4-dimensional non-Abelian gauge theories. It is remarkable that a gauge symmetry can be broken by quantum effects in a similar manner. It should, however, be pointed out that the redundancy associated with the $U(1)$ gauge symmetry of the classical theory still persists at the quantum level. It manifests itself in the $U(1)$ gauge redundancy of the external background lattice gauge fields formed by the self-adjoint extension parameters θ and $\widetilde{\theta}$. In this paper, we have not attempted to take the continuum limit by sending $e^2 a \to 0$. It is conceivable that the full $U(1)$ gauge symmetry would then emerge dynamically from the manifest $\mathbb{Z}(k)$ symmetry, and that the lattice theory would reduce to the non-compact doubled Chern-Simons-Maxwell continuum theory discussed in Section 3.

While the electric field energy does not depend on the values of the self-adjoint extension parameters, the magnetic field energy does. The $\mathbb{Z}(k)$ toric code emerges from the doubled Chern-Simons-Maxwell theory in the limit of infinite "photon" mass, $M = \frac{ke^2}{2\pi} \to \infty$. In this limit the magnetic field energy vanishes, and thus the dependence of the energy on the self-adjoint extension parameters disappears. However, the wave function still depends on θ and $\widetilde{\theta}$. As a result, the toric code has a large variety of hidden self-adjoint extension parameters, which form two $U(1)$ gauge fields, one associated with the original and one associated with the dual lattice. Under adiabatic changes of the parameters θ and $\widetilde{\theta}$, a $U(k)$ Berry gauge field with non-trivial Berry curvature arises. Since the Berry field strength turned out to be gauge equivalent to an Abelian field strength, manipulating the corresponding Berry phases will certainly not allow universal topological quantum computation. Still, it will be interesting to investigate further whether manipulating these parameters can be utilized for other forms of quantum information processing.

In this work, we have encountered a large variety of different gauge fields. First of all, we started out with non-compact Abelian gauge fields A and $\widetilde{A}$, both in the continuum and on the lattice. Upon compactification of the gauge group, this gave rise to the compact $U(1)$ lattice gauge fields φ and $\widetilde{\varphi}$. The group space of an original and dual cross-based link-pair $U(1)^2$ was endowed with an abstract vector potential $(a, \widetilde{a})$ describing the quantized "magnetic" flux represented by the level $k \in \mathbb{Z}$. The Polyakov loops winding around the group space torus $U(1)^2$ then gave rise to two self-adjoint extension parameters θ and $\widetilde{\theta}$, which play the role of Aharonov-Bohm phases for the "mechanical" analog "particle" moving in the group space. While they are invisible at the classical level, at the quantum level the parameters θ and $\widetilde{\theta}$ break the manifest gauge symmetry from $U(1)$ down to $\mathbb{Z}(k)$. Still, the full $U(1)$ redundancy of the gauge description remains, because the self-adjoint extension parameters themselves turned out to be non-dynamical background lattice gauge fields associated with the original and dual lattice, respectively. An extended version of the toric code then emerges at low energies, which is a $\mathbb{Z}(k)$ lattice gauge theory. Finally, by considering adiabatic changes of the self-adjoint extension parameters θ and $\widetilde{\theta}$, $U(k)$ Berry gauge fields arise, which are defined over the base space of the lattice gauge fields θ and $\widetilde{\theta}$. All these different gauge structures are intimately related to one another, and deserve further study, in particular, in the context of quantum information processing.

We have often mentioned the "mechanical" analog of a "particle" moving in the cross-based group space torus $U(1)^2$, threaded by k units of quantized "magnetic" flux. It would be most interesting to build such a system in the laboratory, in order to manipulate quantum information. While building a torus seems like a most difficult task (at least to us), we have great confidence in the ingenuity of AMO experimentalists. Perhaps one can manipulate trapped ions to mimic the physics of the cross-based "mechanical" analog. Using digital quantum simulations with up to 100 quantum gate operations [67], this has already been achieved for a single plaquette of the ordinary (i.e. not extended) $\mathbb{Z}(2)$ toric code [68]. If one could experimentally incorporate the background gauge fields θ and $\widetilde{\theta}$, putting together several crosses would provide us with a remarkable quantum system that embodies a large variety of Abelian dynamical or background gauge fields as well as an additional $U(k)$ Berry connection.

It would be interesting to extend our investigations to non-Abelian Chern-Simons-Maxwell theories on the lattice. In particular, one may ask whether they also reduce to variants of the toric code with a discrete non-Abelian gauge group. We leave this problem for future studies.

Acknowledgments

We like to thank D. Banerjee, M. Blau, M. Dalmonte, M. Lüscher, E. Rico Ortega, A. Smilga, and P. Zoller for illuminating discussions. The research leading to these results has received funding from the Schweizerischer Nationalfonds and from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC grant agreement 339220.

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